(1) Consider the following initial value problem

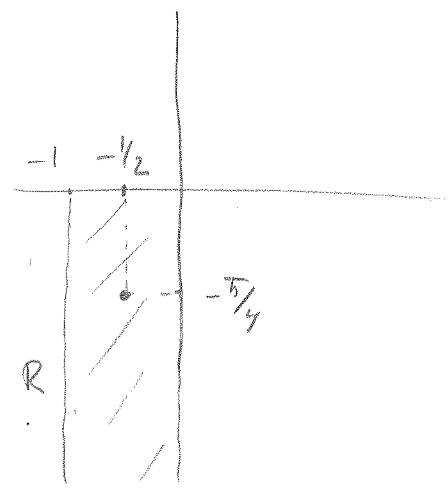
$$\begin{cases} y' = \dfrac{1}{x^2-1} \ln(xy) \\ y(-\tfrac{1}{2}) = -\dfrac{\pi}{4} \end{cases}$$

Using Existence and Uniqueness Theorem, sketch largest possible rectangle about the point $(-\tfrac{1}{2}, -\tfrac{\pi}{4})$ where IVP has a unique solution in.

Put $f(x,y) = \dfrac{1}{x^2-1} \ln(xy)$. If $x<0$ then $y<0$ as well to provide continuity of $f(x,y)$.

Thus on the region $R = \{(x,y) : -1 \le x < 0,\ y<0\}$ the function $f(x,y)$ is continuous. But

$$\frac{\partial f}{\partial y} = \frac{\frac{1}{x^2-1}}{xy} x = \frac{\frac{1}{x^2-1}}{y}$$ is continuous over R.



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a) Characteristic equation: $r^2+2r+1=(r+1)^2$
has roots $-1^{(2)}$. Then

$$y_h = c_1 e^{-t} + c_2 t \cdot e^{-t}$$

$$y_p = t^2(At+B)e^{-t} + C \cdot e^t$$

b) Characteristic equation: $r^4-1=(r^2-1)(r^2+1)$
has roots $\mp 1, \mp i$. Then

$$y_h = c_1 \cdot e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$$

$$y_p = t(A\cos t + B\sin t) + t \cdot (Ct^2+Dt+E)e^t$$

3 a) $$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\, dt = \lim_{A\to\infty} \int_0^A e^{-st} dt = \lim_{A\to\infty}\left(\frac{e^{-st}}{-s} \Big|_0^A \right)$$

$$= \lim_{A\to\infty} \left(\frac{e^{-sA}}{-s} + \frac{1}{s} \right) = \left(0 + \frac{1}{s}\right) = \frac{1}{s}.$$

b) $$1 * t = t * 1 = \int_0^t \tau \cdot 1\, d\tau = \frac{\tau^2}{2}\Big|_0^t = \frac{t^2}{2}$$

c) $$F(s) = \frac{e^{-2s}}{(s-1)^2+1^2} + \frac{s}{(s-1)^2+1^2}$$

$$= e^{-2s} \cdot \frac{1}{(s-1)^2+1^2} + \frac{(s-1)}{(s-1)^2+1^2} + \frac{1}{(s-1)^2+1^2}$$

$$\mathcal{L}^{-1}\{F(s)\} = u_2(t) \cdot e^{(t-2)} \sin(t-2) + e^t \cdot \cos t + e^t \sin t$$

(4)

a) $\det\begin{pmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{pmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda = \mp 1$. (Eigenvalues)

$\lambda = 1$

$$\begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

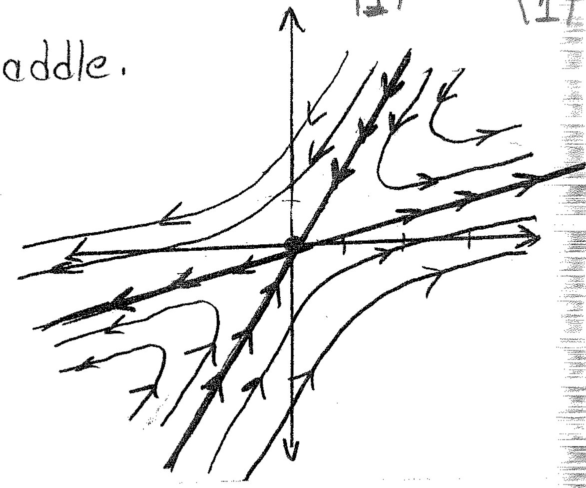
$\lambda = -1$

$$\begin{pmatrix} 3 & -3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = w_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$X(t) = \begin{bmatrix} 3e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$X(t) = c_1 e^{t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

b) $\lambda_1 > 0,\ \lambda_2 < 0$ So, we'll get a saddle.



c) $$X_p(t) = \Psi(t)\cdot\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 3e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$ By using the method of variation of parameters,

we get

$$\Psi(t)\cdot\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 2e^{t} \\ te^{t} \end{bmatrix} \Rightarrow \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \Psi^{-1}(t)\begin{bmatrix} 2e^{t} \\ t\cdot e^{t} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{t} & 3e^{t} \end{bmatrix}\begin{bmatrix} 2e^{t} \\ t\cdot e^{t} \end{bmatrix}$$

$$\Rightarrow u_1' = \frac{1}{2}(2-t) \Rightarrow u_1 = \int \frac{1}{2}(2-t)\,dt = t - \frac{t^2}{4} + c_1$$

$$\Rightarrow u_2' = \frac{1}{2}\left(-2e^{2t} + 3te^{2t}\right) \quad u_2 = \int -e^{2t} + \frac{3}{2}te^{2t}\,dt = -\frac{e^{2t}}{2} + \frac{3}{4}t\cdot e^{2t} - \frac{3}{8}e^{2t} + c_2$$

$$X(t) = \begin{bmatrix} 3e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}\begin{bmatrix} t - \frac{t^2}{4} + c_1 \\ \frac{3}{4}t\cdot e^{2t} - \frac{7}{8}e^{2t} + c_2 \end{bmatrix}$$

(5) $\vec{x}'(t) = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -4 & 3 \\ 1 & -6 & 5 \end{bmatrix} \vec{x}(t)$

Find the fundamental matrix $\Psi(t)$

$$\det(A - tI) = \begin{vmatrix} -1-t & 0 & 0 \\ 1 & -4-t & 3 \\ 1 & -6 & 5-t \end{vmatrix} = -(t+1)\left((t+4)(t-5)+18\right)$$

$$= -(t+1)(t^2-t-2) = -(t+1)^2(t-2)$$

$\sigma(A) = \{-1^{(2)}, 2^{(1)}\}$

$$\lambda = -1 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 3 \\ 1 & -6 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 3 \\ 0 & -3 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = 0, y = z \Rightarrow \xi = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -3 & 3 & 1 \\ 1 & -6 & 6 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -3 & 3 & 1 \\ 0 & -3 & 3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x = 1, y = z \quad \vec{\eta} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = 2 \Rightarrow \begin{bmatrix} -3 & 0 & 0 \\ 1 & -6 & 3 \\ 1 & -6 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 6 & -3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & -3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = 0, 2y = z \Rightarrow \vec{f} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Put $T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \Rightarrow J = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

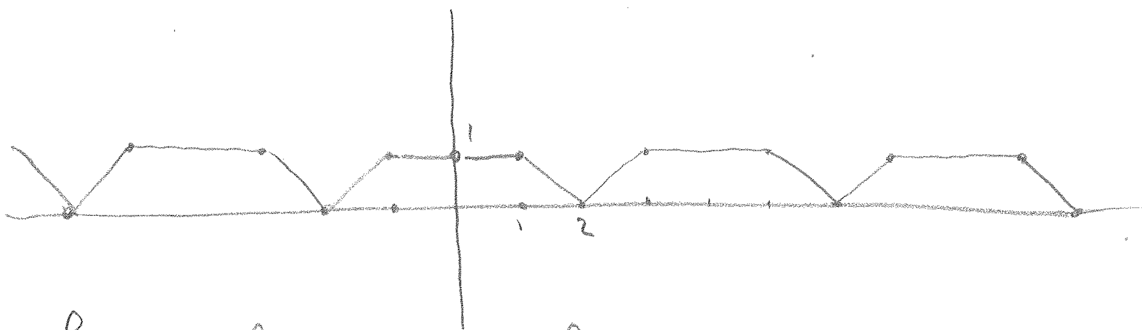
$$e^{Jt} = \begin{bmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

$$\Psi(t) = Te^{Jt} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & e^{-t} & 0 \\ e^{-t} & te^{-t} & e^{2t} \\ e^{-t} & te^{-t} & 2e^{2t} \end{bmatrix}$$

6) Let $f(x) = \begin{cases} 1 & : 0 \le x < 1 \\ 2-x & : 1 \le x < 2 \end{cases}$ be a function defined on the interval $(0, 2)$.

(a) Extend the function to the interval $(-2, 2)$ as an even function and then extend it to the real line as a periodic function (of period 4). Sketch the graph of the obtained function and compute $f(17) = ?$



$f(17) = f(4 \cdot 4 + 1) = f(1) = 1$

(b) Find the Fourier series $S(x)$ of the extended function $f(x)$.

$L = 2$, $a_0 = \int_0^2 f(x)\,dx = 1 + \frac{1}{2} = \frac{3}{2}$, $a_n = \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx$

$= \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 + 2\,\frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\Big|_1^2 - \int_1^2 x \cos\left(\frac{n\pi x}{2}\right) dx$

$= \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) - \frac{4}{n\pi} \sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi} \int_1^2 x \sin'\left(\frac{n\pi x}{2}\right) dx =$

$= \frac{-2}{n\pi} \sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\, x \sin\left(\frac{n\pi x}{2}\right)\Big|_1^2 + \frac{2}{n\pi} \int_1^2 \sin\left(\frac{n\pi x}{2}\right) dx$

$= \frac{-2}{n\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{-2}{n\pi}\,\frac{2}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\Big|_1^2 = \frac{-4}{n^2\pi^2}\left((-1)^n - \cos\left(\frac{n\pi}{2}\right)\right)$

$= \frac{4}{n^2\pi^2}\left(\cos\left(\frac{n\pi}{2}\right) + (-1)^{n+1}\right)$, $b_n = 0$ for all n

Thus $S(x) = \frac{3}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{2}\right) + (-1)^{n+1}}{n^2} \cos\left(\frac{n\pi x}{2}\right)$

(c) Show that the series $S(x)$ converges absolutely for all $x \in \mathbb{R}$.

$$\sum_{n=1}^{\infty} \left| \frac{\cos\left(\frac{n\pi}{2}\right) + (-1)^{n+1}}{n^2} \cos\left(\frac{n\pi x}{2}\right) \right| \leq \sum \frac{2}{n^2} < \infty$$

(d) Compute $S(15)$. Justify your answer.

Since $f(x)$ is a continuous function on $\mathbb{R}$, we conclude that $S(15) = f(15)$ by Fourier Convergence Theorem. But

$f(15) = f(16-1) = f(-1) = 1 \Rightarrow S(15) = 1.$