# METU - NCC

| | Introduction to Differential Equations<br>Midterm 2 | |
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| Code : *MAT 219*<br>Acad.Year: *2012-2013*<br>Semester : *Fall*<br>Date : *29.11.2012*<br>Time : *17:40*<br>Duration : *120 minutes* | Last Name:<br>Name : KEY Student No.:<br>Department: Section:<br>Signature:<br>6 QUESTIONS ON 6 PAGES<br>TOTAL 100 POINTS | |

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**1.** *(7+8=15 pts)* Find the general solution of each differential equation below:

(a) $y^{(4)} - y = 0$.

$$\Delta(t) = t^4 - 1 = (t-1)(t+1)(t-i)(t+i)$$

$$y = c_1 e^{t} + c_2 e^{-t} + c_3 \cos(t) + c_4 \sin(t)$$

(b) $y^{(4)} + y'' = 0$.

$$\Delta(t) = t^4 + t^2 = t^2(t^2+1) = t^2(t-i)(t+i)$$

$$y = c_1 + c_2 t + c_3 \cos(t) + c_4 \sin(t)$$

**2.** *(2+10+3=15 pts)* Consider the differential equation

$$ty'' - 2(t+1)y' + (t+2)y = 0$$

(a) Show that $y_1(t) = e^t$ is a solution.

$$te^t - 2(t+1)e^t + (t+2)e^t = te^t - 2te^t - 2e^t$$
$$+ te^t + 2e^t = 0 \Rightarrow e^t \text{ is a solution}$$

Note that $p(t) = -\frac{2(t+1)}{t}$, $q(t) = \frac{t+2}{t}$, $t \neq 0$.

(b) Find a second solution $y_2$ so that $y_1$ and $y_2$ are independent, and show that they are independent.

Put $y = v(t)e^t$. Then $y' = v'e^t + ve^t$, $y'' =$
$= v''e^t + 2v'e^t + ve^t$, and

$$tv''e^t + 2tv'e^t + tve^t - 2(t+1)v'e^t - 2(t+1)ve^t +$$
$$+ (t+2)ve^t = tv''e^t + (2t-2t-2)v' =$$
$$= tv'' - 2v' = 0$$ (a separable dif. eq. w.r.t. $v'$)

We have $\frac{dv'}{v'} = 2\frac{dt}{t}$ $(v' \neq 0) \Rightarrow v' = ct^2 \Rightarrow$

$$\Rightarrow v = C\tfrac{1}{3}t^3 + C' \Rightarrow y_2 = t^3e^t.$$

$$W(t) = \begin{vmatrix} e^t & t^3e^t \\ e^t & 3t^2e^t + t^3e^t \end{vmatrix} = 3t^2e^{2t} + t^3e^{2t} - t^3e^t$$
$$= 3t^2e^{2t} \neq 0 \quad (t \neq 0) \Rightarrow (y_1, y_2) \text{ are lin. independent solutions.}$$

(c) Write the general solution.

$$y = C_1e^t + C_2t^3e^t$$

**3.** *(8+7=15 pts)* (a) Consider the differential equation $y'' + 3y' - 5y = 0$. Convert it to a first order system of differential equations and show that the characteristic polynomial of the coefficient matrix is $\lambda^2 + 3\lambda - 5$.

Put $x_1 = y$, $x_2 = y'$ and $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

Then we have

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 \\ 5 & -3 \end{bmatrix} \vec{x}(t), \text{ and}$$

$$\Delta(\lambda) = \begin{vmatrix} -\lambda & 1 \\ 5 & -3-\lambda \end{vmatrix} = \lambda(\lambda+3) - 5 =$$

$$= \lambda^2 + 3\lambda - 5.$$

(b) Show that the functions $f(x) = e^{ax}\cos(bx)$ and $g(x) = e^{ax}\sin(bx)$ are linearly independent (where $a, b \in \mathbb{R}$ are constants and $b \neq 0$).

Consider the Wronskian $W(x)$ of these functions:

$$W(x) = \begin{vmatrix} e^{ax}\cos(bx) & e^{ax}\sin(bx) \\ e^{ax}(a\cos(bx) - b\sin(bx)) & e^{ax}(a\sin(bx) + b\cos(bx)) \end{vmatrix}$$

$$= e^{2ax}(a\cos(bx)\sin(bx) + b\cos^2(bx) - a\cos(bx)\sin(bx) + b\sin^2(bx))$$

$$= b\, e^{2ax} \neq 0 \text{ for all } x.$$

Therefore $(f(x), g(x))$ are linearly independent functions from $C(\mathbb{R})$.

**4.** *(8+4+8=20 pts)* Consider the system

$$\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^t$$

(a) Solve the associated homogenous system and find $\mathbf{x}_h$.

$\Delta(\lambda) = \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda-1)(\lambda+1) \Rightarrow \sigma(A) = \{-1^{(1)}, 1^{(1)}\}$

$\lambda = -1 \Rightarrow A + 1 = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \Rightarrow \ker(A+1) = \{3x = y\},\ \vec{f}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda = 1 \Rightarrow A - 1 = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \Rightarrow \ker(A-1) = \{x = y\},\ \vec{f}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Psi(t) = Pe^{Jt} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix} = \begin{bmatrix} e^{-t} & e^t \\ 3e^{-t} & e^t \end{bmatrix}, W(t) = -2$

$\vec{x}(t) = \Psi(t)\vec{c}$ is the general solution.

(b) Find the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = 0$.

$\Phi(t) = e^{At} = Pe^{Jt}P^{-1} = \Psi(t)P^{-1} = \begin{bmatrix} e^{-t} & e^t \\ 3e^{-t} & e^t \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 3/2 & -1/2 \end{bmatrix}$

$= \begin{bmatrix} -\frac{1}{2}e^{-t} + \frac{3}{2}e^t & \frac{1}{2}e^{-t} - \frac{1}{2}e^t \\ -\frac{3}{2}e^{-t} + \frac{3}{2}e^t & \frac{3}{2}e^{-t} - \frac{1}{2}e^t \end{bmatrix}, \quad \Phi(0) = I_2.$

(c) Solve the nonhomogenous system using variation of parameters.

Put $\vec{Y}(t) = \Psi(t)\vec{C}(t)$. Based on VPM, we have

$\Psi(t)\vec{C}'(t) = \vec{b}(t)$ and $c_1'(t) = \frac{1}{-2}\begin{vmatrix} 0 & e^t \\ -e^t & e^t \end{vmatrix} = -\frac{e^{2t}}{2}$,

$c_2'(t) = \frac{1}{-2}\begin{vmatrix} e^{-t} & 0 \\ 3e^{-t} & -e^t \end{vmatrix} = \frac{1}{2}$

Hence $c_1(t) = -\frac{1}{4}e^{2t}$, $c_2(t) = \frac{1}{2}t$, and

$\vec{Y}(t) = \begin{bmatrix} e^{-t} & e^t \\ 3e^{-t} & e^t \end{bmatrix} \begin{bmatrix} -\frac{1}{4}e^{2t} \\ \frac{1}{2}t \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}e^t + \frac{1}{2}te^t \\ -\frac{3}{4}e^t + \frac{1}{2}te^t \end{bmatrix}$. Thus

$\vec{x}(t) = \Psi(t)\vec{C} + \vec{Y}(t)$ is the general solution.

5. *(10+5+5=20 pts)* (a) Consider the initial value problem

$$x' = \begin{bmatrix} -7 & 1 \\ -1 & -5 \end{bmatrix} x', \quad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(a) Find the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = I$ (equivalently $I_2$ or $I_{2\times 2}$).

$$\Delta(\lambda) = \begin{vmatrix} -7-\lambda & 1 \\ -1 & -5-\lambda \end{vmatrix} = (\lambda+7)(\lambda+5)+1 = \lambda^2+12\lambda+36 = (\lambda+6)^2$$

$$\sigma(A) = \{-6^{②}\}, \quad A+6 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow \ker(A+6) = \{x=y\}$$

Thus $V_{-6,1} = \ker(A+6) \subsetneq V_{-6,2} = \ker(A+6)^2 = \mathbb{C}^2, (A+6)^2 = 0.$

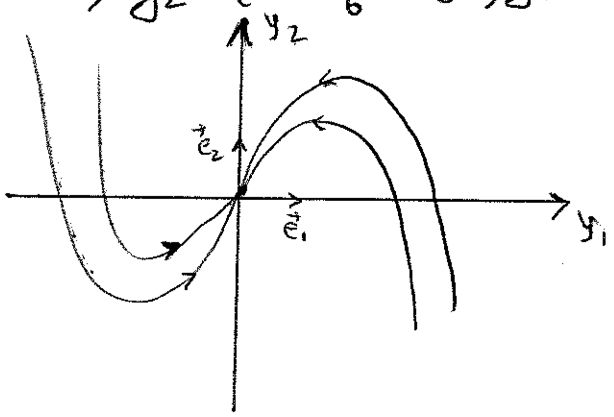
Put $\vec{f}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{f}_2 = (A+6)\vec{f}_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} = P^{-1}$.

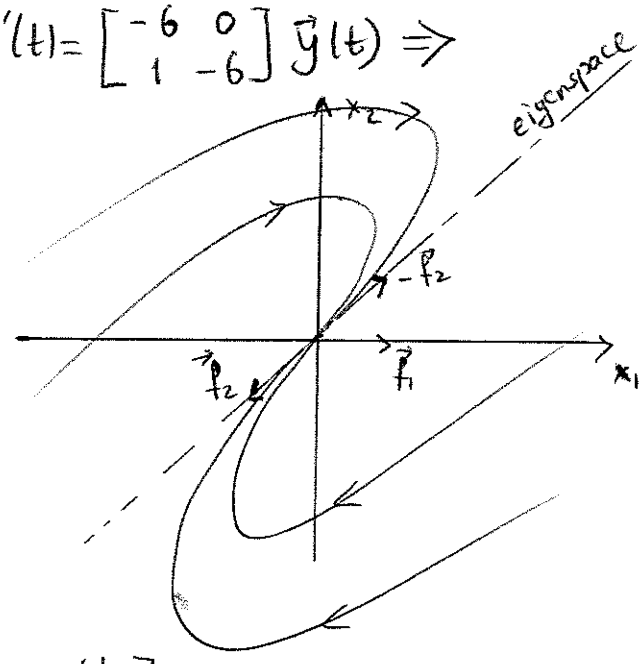
Then $\Psi(t) = Pe^{Jt} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^{-6t} & 0 \\ te^{-6t} & e^{-6t} \end{bmatrix} = \begin{bmatrix} (1-t)e^{-6t} & -e^{-6t} \\ -te^{-6t} & -e^{-6t} \end{bmatrix}$,

$$\Phi(t) = \Psi(t)P^{-1} = \begin{bmatrix} (1-t)e^{-6t} & -e^{-6t} \\ -te^{-6t} & -e^{-6t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} (1-t)e^{-6t} & te^{-6t} \\ -te^{-6t} & (1+t)e^{-6t} \end{bmatrix}$$

(b) Sketch the phase portrait of the system. $\vec{y}'(t) = \begin{bmatrix} -6 & 0 \\ 1 & -6 \end{bmatrix} \vec{y}(t) \Rightarrow$

$\Rightarrow y_2 = (C - \frac{1}{6}\ln|y_1|)y_1$




(c) Solve the initial value problem.

$$\vec{X}(t) = \Phi(t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} (1+t)e^{-6t} \\ (2+t)e^{-6t} \end{bmatrix}$$ is the solution to IVP.

**6.** *(20 pts)* Solve the homogenous system below.

$$x' = \begin{bmatrix} -2 & -1 & -1 \\ 0 & -3 & -1 \\ 0 & 1 & -1 \end{bmatrix} x$$

$$\Delta(\lambda) = \begin{vmatrix} -2-\lambda & -1 & -1 \\ 0 & -3-\lambda & -1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = -(\lambda+2)\left((\lambda+3)(\lambda+1)+1\right) = -(\lambda+2)^3 \Rightarrow \sigma(A) = \{-2^{(3)}\}$$

$$\lambda = -2 \Rightarrow A+2 = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \ker(A+2) = \{y+z=0\},$$

$$m(-2) = 2 < \mathrm{alg}(-2) = 3.$$

$$V_{-2,1} = \ker(A+2) \underset{\neq}{\leq} V_{-2,2} = \ker(A+2)^2 = \mathbb{C}^3$$

Put $\vec{f}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \notin V_{-2,1}$, $\vec{f}_2 = (A+2)\vec{f}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{f}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Thereby $P = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $J = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, and

$$\Psi(t) = Pe^{Jt} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 & 0 \\ te^{-2t} & e^{-2t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix} =$$

$$= \begin{bmatrix} -te^{-2t} & -e^{-2t} & e^{-2t} \\ (1-t)e^{-2t} & -e^{-2t} & 0 \\ te^{-2t} & e^{-2t} & 0 \end{bmatrix}$$ is the fundamental matrix.

In particular, $\vec{x}(t) = \Psi(t)\vec{c}$ is the general solution.