METU
Northern Cyprus Campus

Math 219	Differential Equations	Final Exam	05.06.2011
Last Name: Name : Student No:	Dept./Sec.: Time : 09: 30 Duration : 120 *minutes*	Signature	

6 QUESTIONS ON 4 PAGES — TOTAL 100 POINTS

1	2	3	4	5			
					KEY		

Question 1 (5+10+5=20 p.) Consider the following linear homogeneous differential equation $y'' + y = 0$. Note that $y = C_1\cos(t) + C_2\sin(t)$ is a general solution to $y''+y=0$.

i) Convert it into the linear 2×2-system $\mathbf{x}'(t) = A\mathbf{x}(t)$. Put $x_1 = y$, $x_2 = y'$.

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 \end{aligned} \quad \text{or} \quad \vec{x}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\vec{x}(t).$$

ii) Find the general solution to the linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$.

$$\det(A-\lambda I) = \lambda^2+1 = 0 \Rightarrow \sigma(A) = \{\pm i\}.$$

$$\lambda = i \Rightarrow \left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{\xi} = \begin{bmatrix} 1 \\ i \end{bmatrix} \Rightarrow$$

$$\vec{x}(t) = \vec{\xi} e^{it} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)(\cos(t) + i\sin(t)) =$$

$$= \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + i\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} \Rightarrow \Psi(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

is a fundamental matrix of solutions, $W(t) = \det\Psi(t) = 1$,

$$\vec{x}(t) = \Psi(t)\vec{c} = C_1\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + C_2\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} \quad \text{or}$$

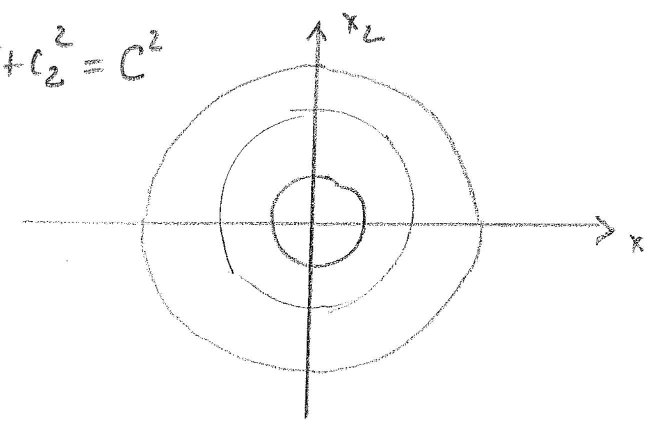
$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad x_1(t) = C_1\cos(t) + C_2\sin(t), \; x_2(t) = -C_1\sin(t) + C_2\cos(t).$$

iii) Sketch the phase portrait of the system and compare with the general solution to the original differential equation.

Note that $x_1(t)^2 + x_2(t)^2 = C_1^2 + C_2^2 = C^2$

But $x_1 = y$ and $x_2 = y' \Rightarrow$

$y(t)^2 + y'(t)^2 = C_1^2 + C_2^2 = C^2$.



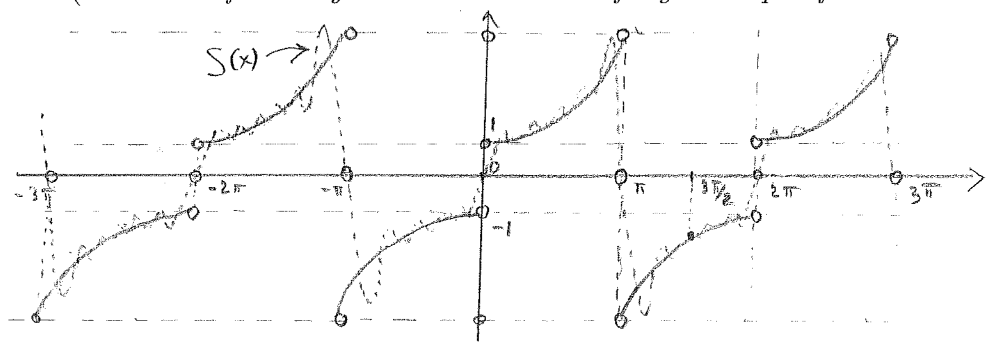
Question 2 (10 p.) Show that the functions $\sin(3x)$ and $\sin(4x)$ are orthogonal on the interval $[-\pi, \pi]$. We have

$$\sin(4x)\cdot\sin(3x) = \int_{-\pi}^{\pi} \sin(4x)\sin(3x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}[\cos(x) - \cos(7x)]dx$$

$$= \int_0^{\pi}[\cos(x) - \cos(7x)]dx = \sin(x)\Big|_0^{\pi} - \frac{1}{7}\sin(7x)\Big|_0^{\pi} =$$

$$= \sin(\pi) - \frac{1}{7}\sin(7\pi) = 0.$$

Question 3 (20 p.) Consider the function $f(x) = e^x$ on the interval $(0, \pi)$. Extend it to the interval $(-\pi, \pi)$ as an odd function, and then to the whole real line as a periodic function. Find its Fourier series $S(x)$, and sketch how does it approximate the original function $f(x)$. Find the values $S(3\pi/2)$ and $S(2\pi)$ based on Fourier Convergence Theorem (*comment: if I were you I would use the shifting techniques from Calculus either*)



Since $f(x)$ is an odd function, we have sine Fourier series $S(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$, where $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx =$

$= \frac{2}{\pi}\int_0^{\pi} e^x \sin(nx)\,dx$. But $\int_0^{\pi} e^x\sin(nx)dx = \frac{-1}{n}\int_0^{\pi} e^x \cos(nx)dx$

$= \frac{-1}{n}e^x\cos(nx)\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi} e^x\cos(nx)dx = \frac{1-e^{\pi}(-1)^n}{n} + \frac{1}{n^2}\int_0^{\pi} e^x \sin(nx)dx$

$= \frac{1+(-1)^{n+1}e^{\pi}}{n} + \underbrace{\frac{1}{n^2}e^x\sin(nx)\Big|_0^{\pi}}_{0} - \frac{1}{n^2}\int_0^{\pi} e^x\sin(nx)dx \Rightarrow$

$\frac{n^2+1}{n^2}\int_0^{\pi} e^x \sin(nx)dx = \frac{1+(-1)^{n+1}e^{\pi}}{n} \Rightarrow b_n = \frac{2}{\pi}\frac{n}{n^2+1}(1+(-1)^{n+1}e^{\pi})$.

Finally, by Fourier Convergence Theorem, $S(x) = \frac{f(x^+)+f(x^-)}{2}$, $\forall x \in \mathbb{R}$. In particular, $S(2\pi) = 0$, and

$S(3\pi/2) = -e^{-(x-2\pi)}\Big|_{x=3\pi/2} = -e^{\pi/2}$ (shifting techniques)

Question 4 (10+15=25 p.) *i)* Find the solution to the following (BVP) boundary-value problem $\begin{cases} y''(t) + y(t) = \cos(t) \\ y(0) = y(\pi/2) = 0 \end{cases}$. Based on MUC, $y = C_1\cos(t) + C_2\sin(t) +$ $+ t(A\cos(t) + B\sin(t))$ (duplication) $\Rightarrow A = 0, B = \frac{1}{2}$, that is, $y(t) = C_1\cos(t) + C_2\sin(t) + \frac{1}{2}t\sin(t)$ – general solution. But $0 = y(0) = C_1$ and $0 = y(\frac{\pi}{2}) =$ $= C_2 + \frac{\pi}{4} \Rightarrow C_2 = -\frac{\pi}{4}$. Hence

$y = -\frac{\pi}{4}\sin(t) + \frac{1}{2}t\sin(t)$ – solution to BVP.

ii) Using the Convolution Theorem for the Laplace transform, find the solution to the following (IVP) initial value problem $\begin{cases} y''(t) + y(t) = \cos(t) \\ y(0) = 0, y'(0) = -\pi/4 \end{cases}$. Show that the indicated BVP and IVP have exactly the same solutions. Hint: *Compute the relevant convolution based on its precise definition.*

Put $Y(s) = \mathcal{L}\{y(t)\}$. Then $\mathcal{L}\{y''(t)\} = s^2 Y(s) + \frac{\pi}{4}$ and

$(s^2+1)Y(s) + \frac{\pi}{4} = \mathcal{L}\{\cos(t)\} \Rightarrow Y(s) = -\frac{\pi}{4}\frac{1}{s^2+1} + \frac{1}{s^2+1}\mathcal{L}\{\cos(t)\}$

Hence $y(t) = -\frac{\pi}{4}\sin(t) + \sin(t) * \cos(t)$.

Further, $\sin(t) * \cos(t) = \int_0^t \sin(t-\tau)\cos(\tau)\,d\tau =$

$= \int_0^t [\sin(t)\cos^2(\tau) - \cos(t)\sin(\tau)\cos(\tau)]\,d\tau =$

$= \sin(t)\int_0^t \cos^2(\tau)\,d\tau - \frac{\cos(t)}{2}\int_0^t \sin(2\tau)\,d\tau =$

$= \frac{\sin(t)}{2}\left(t + \frac{1}{2}\sin(2\tau)\Big|_0^t\right) + \frac{\cos(t)}{4}\cos(2\tau)\Big|_0^t =$

$= \frac{t\sin(t)}{2} + \frac{1}{4}\underbrace{(\cos(2t)\cos(t) + \sin(2t)\sin(t))}_{\cos(2t-t)} - \frac{\cos(t)}{4}$

$= \frac{1}{2}t\sin(t)$.

Thus we have got the same solution

$y(t) = -\frac{\pi}{4}\sin(t) + \frac{1}{2}t\sin(t)$.

Question 5 (5+5+10+5=25 p.) Let $A = \begin{bmatrix} -3 & -2 & 1 \\ 0 & -3 & 5 \\ 0 & 0 & -3 \end{bmatrix} \in M_3$.

i) Find its Jordan matrix J. Note that $\sigma(A) = \{-3^{(3)}\}$. The generalized eigenvectors

$$\vec{\xi} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{\eta} = \begin{bmatrix} 0 \\ -1/2 \\ 0 \end{bmatrix}, \quad \vec{\theta} = \begin{bmatrix} 0 \\ 0 \\ -1/10 \end{bmatrix} \Rightarrow J = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix} = T^{-1} A T,$$

where $T = [\vec{\xi}\ \vec{\eta}\ \vec{\theta}]$.

ii) Find A^{101} using J.

$$J^{101} = \begin{bmatrix} -3^{101} & 101 \cdot 3^{100} & -5050 \cdot 3^{99} \\ 0 & -3^{101} & 101 \cdot 3^{100} \\ 0 & 0 & -3^{101} \end{bmatrix} \Rightarrow A^{101} = T J^{101} T^{-1} = \begin{bmatrix} -3^{101} & -202 \cdot 3^{100} & 50500 \cdot 3^{99} \\ 0 & -3^{101} & 505 \cdot 3^{100} \\ 0 & 0 & -3^{101} \end{bmatrix}$$

iii) Find the fundamental matrix of solutions to the linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$.

$$\Psi(t) = T e^{Jt} = e^{-3t} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/10 \end{bmatrix} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= e^{-3t} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & -1/2 & -t/2 \\ 0 & 0 & -1/10 \end{bmatrix}$$

$\vec{x}(t) = \Psi(t)\vec{c}$ – general solution to the system.

iv) Find e^A. Note that $\Phi(t) = \Psi(t) T^{-1} =$

$$= e^{-3t} \begin{bmatrix} 1 & -2t & -5t^2 \\ 0 & 1 & 5t \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow e^A = \Phi(1) = e^{-3} \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$