

M E T U
Northern Cyprus Campus

Math 210 Applied Mathematics for Engineers					Final Exam	03.06.2010
Last Name:	Dept./Sec. :			Signature		
Name :	Time : 13:00					
Student No: SOLUTIONS	Duration : 110 minutes					
4 QUESTIONS ON 4 PAGES					TOTAL 100 POINTS	
1	2	3	4			

25p Q1 In the following two parts, A , w_1 , w_2 , and v are as follows:

$$A = \begin{bmatrix} 2 & 0 & -1 & 0 & -3 \\ -4 & 0 & 3 & 1 & 1 \\ 2 & 0 & -1 & 0 & -3 \\ -2 & -2 & 3 & 3 & -1 \\ 2 & 0 & -1 & 0 & -3 \end{bmatrix} \quad w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

The vector v is an eigenvector of A . One of w_1 , w_2 is an eigenvector but the other isn't.

(i) Which one of w_1 and w_2 is the eigenvector? Show your work. Check \bar{w}_1 and \bar{w}_2 .

$$\begin{bmatrix} 2 & 0 & -1 & 0 & -3 \\ -4 & 0 & 3 & 1 & 1 \\ 2 & 0 & -1 & 0 & -3 \\ -2 & -2 & 3 & 3 & -1 \\ 2 & 0 & -1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -3 \\ -4 + 3 + 1 \\ 2 & -1 & -3 \\ -2 + 3 & -1 \\ 2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ -2 \end{bmatrix} = (-2) \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \textcircled{8}$$

$$\begin{bmatrix} 2 & 0 & -1 & 0 & -3 \\ -4 & 0 & 3 & 1 & 1 \\ 2 & 0 & -1 & 0 & -3 \\ -2 & -2 & 3 & 3 & -1 \\ 2 & 0 & -1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 + 0 - 3 \\ 0 + 1 + 1 \\ 0 + 0 - 3 \\ -2 + 3 - 1 \\ 0 + 0 - 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ -3 \\ 0 \\ -3 \end{bmatrix} \quad \textcircled{8}$$

no part 2

w_1 is an eigenvector.

$(w_2 \text{ is not})$

(ii) What is the eigenvalue for the eigenvector v ? Show your work.

$$\begin{bmatrix} 2 & 0 & -1 & 0 & -3 \\ -4 & 0 & 3 & 1 & 1 \\ 2 & 0 & -1 & 0 & -3 \\ -2 & -2 & 3 & 3 & -1 \\ 2 & 0 & -1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 + 0 \\ 0 + 2 \\ 0 + 0 \\ -2 + 6 \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 4 \\ 0 \end{bmatrix} = (2) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \quad \textcircled{9}$$

eigenvalue is 2

25 p^{ts}Q2 For the function $f(x, y, z) = xy - xz^2$ calculate the following:(a) ∇f

$$\nabla f = \left(\frac{\partial}{\partial x} (xy - xz^2), \frac{\partial}{\partial y} (xy - xz^2), \frac{\partial}{\partial z} (xy - xz^2) \right) \quad 3 \text{ pts}$$

$$\stackrel{(5)}{=} \boxed{\begin{pmatrix} y - z^2, & x, & -2xz \end{pmatrix}} \quad 2 \text{ pts}$$

(b) The directional derivative $D_{\mathbf{v}}f(1, 0, 1)$ in the direction $\mathbf{v} = [\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}]$

$$\begin{aligned} D_{\mathbf{v}}f(1, 0, 1) &= \mathbf{v} \cdot \nabla f(1, 0, 1) \quad \left(\text{Note: } |\mathbf{v}| = \sqrt{\frac{1+4+4}{9}} = 1 \right) \\ &= \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right) \cdot (-1, 1, -2) \\ &= -\frac{1}{3} + \frac{2}{3} + \frac{4}{3} = \boxed{\frac{5}{3}} \end{aligned} \quad 3 \text{ pts}$$

(c) The normal vector of $f = 1$ at the point $(1, 0, 1)$.

$$\begin{aligned} \stackrel{(5)}{\bar{n}} &= \nabla f(1, 0, 1) \\ &= \boxed{(-1, 1, -2)} \quad \xrightarrow{\text{normalize (optional)}} \boxed{\frac{1}{\sqrt{6}}(-1, 1, -2)} \end{aligned} \quad 3 \text{ pts}$$

(d) $\text{Curl}(\nabla f)$

$$\text{Curl}(\nabla f) = \nabla \times (\nabla f) = \nabla \times (y - z^2, x, -2xz) \quad 3 \text{ pts}$$

$$\begin{aligned} \stackrel{(5)}{\text{Note:}} \quad \text{Curl}(\nabla f) &= \nabla \times \nabla f \\ &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z^2 & x & -2xz \end{vmatrix} \\ &= (0 - 0, -(-2z - (-2z)), 1 - 1) \end{aligned}$$

(e) $\text{Div}(\nabla f)$

$$\text{Div}(\nabla f) = \frac{\partial}{\partial x}(y - z^2) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(-2xz) \quad 3 \text{ pts}$$

$$\begin{aligned} \stackrel{(5)}{=} & 0 + 0 + (-2x) \\ & = \boxed{-2x} \end{aligned} \quad 2 \text{ pts}$$

25 pts

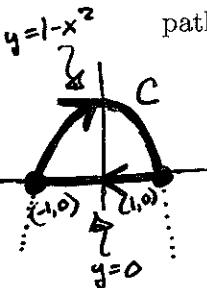
Q3 Solve the following 2D integral calculus problems:

(a) Write Green's theorem for the circulation of a vector field \mathbf{F} around a closed loop C .

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$$

(circulation)

$$\left. \begin{aligned} & \oint_C \mathbf{F} \cdot d\mathbf{r} \\ & \oint_C P dx + Q dy \end{aligned} \right\} \text{aka. } = \iint_{\text{inside of } C} \left(\frac{\partial}{\partial x} Q - \frac{\partial}{\partial y} P \right) dx dy \quad 5 \text{ pts}$$

(b) Use a line integral to compute the circulation ($\oint_C \mathbf{F} \cdot d\mathbf{r}$) of $\mathbf{F} = xy\mathbf{i} + x\mathbf{j}$ around the closed path given by $y = 1 - x^2$ from $(-1, 0)$ to $(1, 0)$ followed by the line from $(1, 0)$ back to $(-1, 0)$.

$$\begin{aligned} \oint_C xy \, dx + x \, dy &= \int_A xy \, dx + x \, dy + \int_B xy \, dx + x \, dy \\ &\quad \text{A: } \begin{cases} x=t & dx=dt \\ y=1-t^2 & dy=-2t \, dt \\ -1 \leq t \leq 1 \end{cases} \quad \text{B: } \begin{cases} x=-t & dx=-dt \\ y=0 & dy=0 \\ -1 \leq t \leq 1 \end{cases} \\ &= \int_{-1}^1 t(1-t^2) + t(-2t) \, dt + \int_{-1}^1 (-t) \cdot 0 + (-t) \cdot 0 \, dt \\ &= \frac{1}{2}t^2 - \frac{1}{4}t^4 - \frac{2}{3}t^3 \Big|_{t=-1}^{t=1} + 0 \quad 10 \text{ pts} \\ &= -\frac{2}{3} - \frac{2}{3} = \boxed{-\frac{4}{3}} \end{aligned}$$

(c) Use Green's theorem to compute the same circulation using a double integral.

$$\begin{aligned} \oint_C xy \, dx + x \, dy &= - \iint_R \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x) \right) dA \\ &\quad \text{Note: Minus sign b/c we do the loop clockwise} \\ &= - \int_{x=-1}^{x=1} \int_{y=0}^{y=1-x^2} (1-x) \, dy \, dx \quad 10 \text{ pts} \\ &= - \int_{-1}^1 (1-x)(1-x^2) \, dx \\ &= - \left(x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 \Big|_{x=-1}^{x=1} \right) \\ &= - (2 - \frac{2}{3}) \\ &= \boxed{-\frac{4}{3}} \end{aligned}$$

25 pts Q4 Solve the following 3D integral calculus problems:

(a) Write the divergence theorem. State in words what it means.

$$E = P\hat{i} + Q\hat{j} + R\hat{k}$$

$$\oint_S (\text{flux}) \oint_S \vec{F} \cdot \hat{n} \, d\sigma \quad \left\{ \begin{array}{l} \text{aka.} \\ \oint_S P \, dy \, dz + Q \, dz \, dx + R \, dx \, dy \end{array} \right\} = \left\{ \begin{array}{l} \iiint_V \text{Div}(\vec{F}) \, dV \\ \text{aka.} \\ \iiint_V \frac{\partial}{\partial x} P + \frac{\partial}{\partial y} Q + \frac{\partial}{\partial z} R \, dx \, dy \, dz \end{array} \right\}$$

10 pts

|| The flux of a vector field across a (closed, smooth, oriented) surface is equal to the integral over the inside of the surface of the divergence of the vector field (the "infinitesimal flux" at each point).

(b) Show that the flux of the vector field $\vec{F} = (x+y)\hat{i} + (z+x)\hat{j} + (x-y)\hat{k}$ across any closed, smooth oriented surface S is equal to the volume enclosed by the surface. (Show all your work and reasoning.)

$$\begin{aligned} \text{Flux} &= \oint_S (x+y) \, dy \, dz + (z+x) \, dz \, dx + (x-y) \, dx \, dy \\ &= \iiint_V \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(z+x) + \frac{\partial}{\partial z}(x-y) \, dx \, dy \, dz \quad 10 \text{ pts} \\ &\stackrel{\text{Divergence Thm.}}{=} \iiint_V 1 + 0 + 0 \, dx \, dy \, dz \\ &= \iiint_V 1 \, dx \, dy \, dz \\ &= \text{Volume}(V) \quad 5 \text{ pts} \\ &\text{The volume enclosed by } S. \end{aligned}$$