

Math 210 Applied Mathematics for Engineers II. Exam 28.04.2010				
Last Name: <u>SOLUTIONS</u>		Dept./Sec. :		Signature
Name :		Time : 17:40		
Student No: _____		Duration : 110 minutes		
5 QUESTIONS ON 6 PAGES				TOTAL 100 POINTS
1	2	3	4	5

Q1 (5+5=10 pts.) Consider the space transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which reflects points across the $y = x$ plane in space.

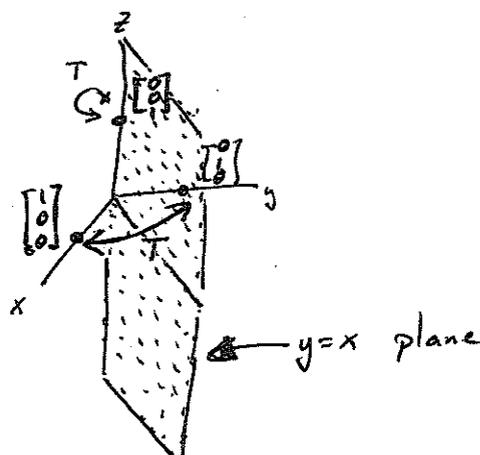
(i) Write T as a matrix (using the standard basis).

$$A = \begin{bmatrix} | & | & | \\ T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



(ii) Is this transformation invertible? Justify your answer.

Yes. The matrix of the transformation $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has $\det = -1 \neq 0$. So it is invertible.

(The inverse of T is just to reflect back across the $y=x$ plane.)

Q2 (10+5=15 pts.) In the following two parts, A, w_1 , w_2 , and v are as follows:

$$A = \begin{bmatrix} 4 & 6 & -2 & -4 & -2 \\ 6 & 4 & 0 & -2 & -6 \\ 0 & 0 & 2 & 2 & -2 \\ 6 & 6 & 0 & -4 & -6 \\ 8 & 8 & -2 & -6 & -6 \end{bmatrix} \quad w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad v = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The vector v is an eigenvector of A . One of w_1 , w_2 is an eigenvector but the other isn't.

(i) Which one of w_1 and w_2 is the eigenvector? Show your work.

$$Aw_1 = \begin{bmatrix} 4 & 6 & -2 & -4 & -2 \\ 6 & 4 & 0 & -2 & -6 \\ 0 & 0 & 2 & 2 & -2 \\ 6 & 6 & 0 & -4 & -6 \\ 8 & 8 & -2 & -6 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4-2 \\ 6+0 \\ 0+2 \\ 6+0 \\ 8-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \\ 6 \\ 6 \end{bmatrix} \neq 1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 1 w_1$$

$$Aw_2 = \begin{bmatrix} 4 & 6 & -2 & -4 & -2 \\ 6 & 4 & 0 & -2 & -6 \\ 0 & 0 & 2 & 2 & -2 \\ 6 & 6 & 0 & -4 & -6 \\ 8 & 8 & -2 & -6 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6-4-2 \\ 4-2-6 \\ 0+2-2 \\ 6-4-6 \\ 8-6-6 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 0 \\ -4 \\ -4 \end{bmatrix} = (-4) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = (-4) w_2$$

w_2 is an eigenvector

(ii) What is the eigenvalue for the eigenvector v ? Show your work.

$$Av = \begin{bmatrix} 4 & 6 & -2 & -4 & -2 \\ 6 & 4 & 0 & -2 & -6 \\ 0 & 0 & 2 & 2 & -2 \\ 6 & 6 & 0 & -4 & -6 \\ 8 & 8 & -2 & -6 & -6 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4+6 \\ -6+4 \\ 0+0 \\ -6+6 \\ -8+8 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = (-2) \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -2 v$$

eigenvalue = -2

Q3 (10+10+10=30 pts.) The following questions deal with a matrix A which has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 1$ and $\lambda_3 = -2$ and associated eigenvectors:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(Solve these questions without computing A .)

(i) What is the determinant of A ?

$$\begin{aligned} A &= P D P^{-1} \\ \det(A) &= \det(P D P^{-1}) \\ &= (\det P)(\det D)(\det P^{-1}) \\ &= (\cancel{\det P})(\det D) \frac{1}{(\cancel{\det P})} \\ &= \det D = 2 \cdot 1 \cdot (-2) = \boxed{-4} \end{aligned}$$

(ii) Is A symmetric? Justify your answer.

$$\begin{aligned} v_1 \cdot v_2 &= 1 \cdot (-1) + 1 \cdot (0) + 1 \cdot (1) = 0 & \text{so } v_1 \perp v_2 \\ v_1 \cdot v_3 &= 1 \cdot (0) + 1 \cdot (1) + 1 \cdot (-1) = 0 & \text{so } v_1 \perp v_3 \\ v_2 \cdot v_3 &= (-1) \cdot 0 + 0 \cdot 1 + 1 \cdot (-1) = -1 & \text{so } v_2 \not\perp v_3 \end{aligned}$$

A is not symmetric because v_2 and v_3 have different eigenvalues but are not orthogonal

(iii) What are the eigenvalues and eigenvectors of A^2 ?

$$\begin{aligned} A^2 &= A \cdot A = P D P^{-1} \cdot P D P^{-1} \\ &= P D^2 P^{-1} \end{aligned}$$

eigenvalues	$\lambda_1 = 2^2$	$\lambda_2 = 1^2$	$\lambda_3 = (-2)^2$
eigenvectors	$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$	$v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

(iv) What is the solution to $Ax = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$?

$$\begin{aligned} x &= A^{-1} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \\ &= A^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot 3 \\ &= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot 3 = \begin{bmatrix} 3/2 \\ 3/2 \\ 3/2 \end{bmatrix} \end{aligned}$$

Note: A^{-1} has eigenvalues/eigenvectors
 $\lambda = 1/2, 1/1, 1/(-2)$
 $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Q4 (5+15+10=30 pts.) The following parts deal with computations of eigenvalues and eigenvectors.

(i) Find the eigenvalues of $\begin{bmatrix} 2 & -2 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$. (Do not find eigenvectors.)

characteristic eqn: $0 = \det \begin{bmatrix} 2-\lambda & -2 & 0 \\ -1 & 1-\lambda & 0 \\ -1 & 1 & -\lambda \end{bmatrix}$

(expand down last col) $0 = (-\lambda) \det \begin{bmatrix} 2-\lambda & -2 \\ -1 & 1-\lambda \end{bmatrix}$

$$0 = (-\lambda)(\lambda^2 - 2\lambda)$$

$$0 = -\lambda^2(\lambda - 2)$$

$\lambda = 0, 0, 2$

(Note: The matrix clearly has rank 1, so 0 should be an eigenval of multiplicity 2.)

(ii) The matrix $\begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ has eigenvalues $\lambda = \cancel{1}, \cancel{2}, 0$. Find associated eigenvectors.

What is unusual about your result?

It is unusual that the person who wrote this exam incorrectly said that 1 is an eigenvalue. If the problem had stated 2 was an eigenvalue, then it would have been unusual that algebraic mult \neq geometric mult.

0-eigenspace:

$$\left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 2 & 2 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} x_1 + x_2 = 0 \\ x_3 = 0 \end{array}$$

$$x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} x_2$$

$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

1-eigenspace:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

1 is not an eigenvalue.

Corrected problem:

2-eigenspace:

$$\left[\begin{array}{ccc|c} 0 & 2 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & -1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(iii) Compute the matrix A with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 0$ and $\lambda_3 = 1$ and associated eigenvectors

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

(Write your answer as a matrix – not as a product of matrices.)

$$A = P D P^{-1}$$

Compute P, D, P^{-1} :

$$P = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \quad \Bigg| \quad D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \quad \Bigg| \quad = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: P is orthogonal, so $P^{-1} = P^T$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \frac{1}{2} & 0 & 1 - \frac{1}{2} \\ 0 & 0 & 0 \\ 1 - \frac{1}{2} & 0 & 1 + \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 3/2 \end{bmatrix}$$

Note: A is symmetric, as it should be since eigenvalues are real and eigenvectors are \perp .

Q5 (5+10=15 pts.) A is a real, symmetric matrix with eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad v_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

(i) Are the vectors v_1 , v_2 , and v_3 all orthogonal to each other?

$$v_1 \cdot v_2 = 1 \cdot (-1) + 0 \cdot 1 + 1 \cdot 1 = 0 \quad \text{so} \quad v_1 \perp v_2$$

$$v_1 \cdot v_3 = 1 \cdot (-1) + 2 \cdot 0 + 1 \cdot 1 = 0 \quad \text{so} \quad v_1 \perp v_3$$

$$v_2 \cdot v_3 = (-1)(-1) + 1 \cdot 2 + 1 \cdot 1 = 4 \quad \text{so} \quad v_2 \not\perp v_3$$

Not all orthogonal.

(ii) What can you say about the eigenvalues associated to v_1 , v_2 , and v_3 ?

(Note: There is no mistake in this problem: There are real, symmetric matrices with these eigenvectors.)

• The eigenvalues are all real because

A is real, symmetric.

• The eigenvalues for v_2 and v_3 are equal

($\lambda_2 = \lambda_3$) because $v_2 \not\perp v_3$.