

Math 210 Applied Mathematics for Engineers I. Exam 24.03.2010			
Last Name: Name: SOLUTION Student No:	Dept./Sec. : Time : 17:40 Duration : 110 minutes	Signature	
4 QUESTIONS ON 4 PAGES			TOTAL 100 POINTS
1	2	3	4

Q1 (5+5+5+5=20 pts.) Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}.$$

Compute the following (if defined):

(i) BA.

$$\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \text{ is } \underline{\underline{\text{not defined.}}}$$

$$\begin{matrix} 2 \times \boxed{2} & \boxed{3} \times 2 \\ \uparrow & \uparrow \\ \text{do not} & \\ \text{match} & \end{matrix}$$

(ii) 2B + C.

$$\begin{aligned} 2 \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} &= \begin{bmatrix} -2 & 2 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

(iii) A^T.

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

(iv) (AB)^T.

$$\begin{aligned} \left(\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \right)^T &= \begin{bmatrix} -1+2 & 1+0 \\ 0-1 & 0+0 \\ -1+1 & 1+0 \end{bmatrix}^T \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{aligned}$$

Q2 (5+5+5+5+5+5=30 pts.) Consider the matrix equation $Ax = b$ given by

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix}$$

(i) Write the matrix equation as a system of equations and as an augmented matrix.

$$\begin{array}{l} x_2 + 2x_3 = 4 \\ x_1 + x_3 = 4 \\ x_1 + 2x_3 = 5 \end{array} \quad \left[\begin{array}{ccc|c} 0 & 1 & 2 & 4 \\ 1 & 0 & 1 & 4 \\ 1 & 0 & 2 & 5 \end{array} \right]$$

(ii) Compute the determinant of the matrix A using row and/or column expansion.

$$\det \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} = -1 \cdot \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} - 0 \cdot \det \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

(Expand down column 2) $= - (1 \cdot 2 - 1 \cdot 1) = \boxed{-1}$

(iii) Find the solution to the matrix equation using row reduction.

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 4 \\ 1 & 0 & 1 & 4 \\ 1 & 0 & 2 & 5 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 1 & 0 & 2 & 5 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$x_1 + x_3 = 4 \Rightarrow x_1 = 3$
 $x_2 + 2x_3 = 4 \Rightarrow x_2 = 2$
 $x_3 = 1$

$$\bar{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

(iv) Find the inverse of the matrix A (by any method).

Using Gauss-Jordan reduction:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right]$$

(students could also use the adjoint formula: $A^{-1} = \frac{1}{\det A} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^T$)

$$A^{-1} = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

(v) Find the solution to the matrix equation using the inverse matrix from (iv).

$$\bar{x} = A^{-1} \bar{b} = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 8-5 \\ 4+8-10 \\ -4+5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

(vi) Find x_3 using Cramer's rule.

$$x_3 = \frac{\det A_3}{\det A} = \frac{\det \begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 4 \\ 1 & 0 & 5 \end{bmatrix}}{\det \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}} = \frac{-1 \cdot \det \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}}{-1 \cdot \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}} = \frac{-1}{-1}$$

$$\boxed{x_3 = 1}$$

Q3 (6+6+6+6+6+6=36 pts.)

The matrix A reduces as indicated below:

$$A = \begin{bmatrix} 2 & 1 & 3 & 2 \\ 4 & 1 & 8 & 5 \\ -2 & -3 & 1 & -2 \\ 8 & 2 & 16 & 16 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 3 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(i) What is the rank of A? Why?

Rank(A) = 3, because A reduces to have 3 pivots
(equivalently, A reduces to have 3 nonzero rows)

(ii) What is the nullity of A (i.e. the dimension of the nullspace)? Why?

Nullity(A) = 1, because Rank(A) + Nullity(A) = # columns
(equivalently, one column of A reduces to have no pivot so it will give a free variable)

(iii) Are the rows of A linearly independent? Why?

No, because there are 4 rows but Rank(A) = 3
(equivalently, one row of A reduces to be all zeros; or, one row of A does not add a pivot to ref(A))

(iv) Is there a b so that Ax = b has no solution? Why?

Yes, because Rank(A) < # rows
(equivalently, one row of ref(A) is all zero, so some b will give a nonzero entry in that row after reduction, leading to a contradiction: 0 = *)

(v) Give the solution to Ax = 0.

$$\left[\begin{array}{cccc|c} 2 & 1 & 3 & 2 & 0 \\ 4 & 1 & 8 & 5 & 0 \\ -2 & -3 & 1 & -2 & 0 \\ 8 & 2 & 16 & 16 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 2 & 1 & 3 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} 2x_1 + x_2 + 3x_3 + 2x_4 = 0 \implies x_1 = \frac{5}{2}x_3 \\ -x_2 + 2x_3 + x_4 = 0 \implies x_2 = 2x_3 \\ -2x_4 = 0 \\ 0 = 0 \implies x_4 = 0 \end{array}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{5}{2}x_3 \\ 2x_3 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 2 \\ 1 \\ 0 \end{bmatrix} x_3$$

(vi) Ax = $\begin{bmatrix} 4 \\ 4 \\ -10 \\ 2 \end{bmatrix}$ has a solution x = $\begin{bmatrix} 4 \\ 1 \\ -1 \\ -1 \end{bmatrix}$. Give another solution.

(Hint: this should not require algebra.)

The general solution is given by $\vec{x} = \vec{x}_p + \vec{x}_h$

So $\vec{x} = \begin{bmatrix} 4 \\ 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} \frac{5}{2} \\ 2 \\ 1 \\ 0 \end{bmatrix} a$ are all solutions.

Q4 (14 pts.) Find the determinant of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

Using row reduction:

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \xrightarrow{\substack{r_5 \rightarrow r_4 \\ r_4 \rightarrow r_3}} \det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{r_4 \rightarrow r_3} \det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{r_3 \rightarrow r_2} \det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{r_2 \rightarrow r_1} \det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \boxed{1}$$

(It is also possible, though very difficult, to find the determinant using row/column expansions.)