EXAMPLE 5 If $R = [0, \pi/2] \times [0, \pi/2]$, then, by Equation 5,

$$\iint_R \sin x \cos y \, dA = \int_0^{\pi/2} \sin x \, dx \int_0^{\pi/2} \cos y \, dy$$

$$= \Big[-\cos x\Big]_0^{\pi/2} \Big[\sin y\Big]_0^{\pi/2} = 1 \cdot 1 = 1$$

The function $f(x, y) = \sin x \cos y$ in Example 5 is positive on R, so the integral represents the volume of the solid that lies above R and below the graph of f shown in Figure 6.

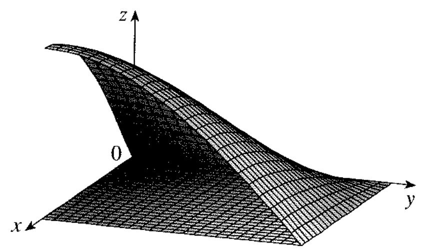

FIGURE 6

15.2 Exercises

1–2 Find $\int_0^5 f(x, y)\, dx$ and $\int_0^1 f(x, y)\, dy$.

1. $f(x, y) = 12x^2y^3$

2. $f(x, y) = y + xe^y$

3–14 Calculate the iterated integral.

3. $\int_1^4 \int_0^2 (6x^2y - 2x)\, dy\, dx$

4. $\int_0^1 \int_1^2 (4x^3 - 9x^2y^2)\, dy\, dx$

5. $\int_0^2 \int_0^{\pi/2} x \sin y \, dy\, dx$

6. $\int_{\pi/6}^{\pi/2} \int_{-1}^5 \cos y \, dx\, dy$

7. $\int_{-3}^3 \int_0^{\pi/2} (y + y^2 \cos x)\, dx\, dy$

8. $\int_0^1 \int_1^2 \dfrac{xe^x}{y}\, dy\, dx$

9. $\int_1^4 \int_1^2 \left(\dfrac{x}{y} + \dfrac{y}{x}\right) dy\, dx$

10. $\int_0^1 \int_0^3 e^{x+3y}\, dx\, dy$

11. $\int_0^1 \int_0^1 v(u + v^2)^4\, du\, dv$

12. $\int_0^1 \int_0^1 xy\sqrt{x^2 + y^2}\, dy\, dx$

13. $\int_0^2 \int_0^\pi r \sin^2\theta \, d\theta\, dr$

14. $\int_0^1 \int_0^1 \sqrt{s + t}\, ds\, dt$

15–22 Calculate the double integral.

15. $\iint_R \sin(x - y)\, dA, \quad R = \{(x, y) \mid 0 \le x \le \pi/2, 0 \le y \le \pi/2\}$

16. $\iint_R (y + xy^{-2})\, dA, \quad R = \{(x, y) \mid 0 \le x \le 2, 1 \le y \le 2\}$

17. $\iint_R \dfrac{xy^2}{x^2 + 1}\, dA, \quad R = \{(x, y) \mid 0 \le x \le 1, -3 \le y \le 3\}$

18. $\iint_R \dfrac{1 + x^2}{1 + y^2}\, dA, \quad R = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$

19. $\iint_R x \sin(x + y)\, dA, \quad R = [0, \pi/6] \times [0, \pi/3]$

20. $\iint_R \dfrac{x}{1 + xy}\, dA, \quad R = [0, 1] \times [0, 1]$

21. $\iint_R ye^{-xy}\, dA, \quad R = [0, 2] \times [0, 3]$

22. $\iint_R \dfrac{1}{1 + x + y}\, dA, \quad R = [1, 3] \times [1, 2]$

23–24 Sketch the solid whose volume is given by the iterated integral.

23. $\int_0^1 \int_0^1 (4 - x - 2y)\, dx\, dy$

24. $\int_0^1 \int_0^1 (2 - x^2 - y^2)\, dy\, dx$

25. Find the volume of the solid that lies under the plane $4x + 6y - 2z + 15 = 0$ and above the rectangle $R = \{(x, y) \mid -1 \le x \le 2, -1 \le y \le 1\}$.

26. Find the volume of the solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$.

27. Find the volume of the solid lying under the elliptic paraboloid $x^2/4 + y^2/9 + z = 1$ and above the rectangle $R = [-1, 1] \times [-2, 2]$.

28. Find the volume of the solid enclosed by the surface $z = 1 + e^x \sin y$ and the planes $x = \pm 1$, $y = 0$, $y = \pi$, and $z = 0$.

29. Find the volume of the solid enclosed by the surface $z = x \sec^2 y$ and the planes $z = 0$, $x = 0$, $x = 2$, $y = 0$, and $y = \pi/4$.

30. Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 5$.

31. Find the volume of the solid enclosed by the paraboloid $z = 2 + x^2 + (y - 2)^2$ and the planes $z = 1$, $x = 1$, $x = -1$, $y = 0$, and $y = 4$.

32. Graph the solid that lies between the surface $z = 2xy/(x^2 + 1)$ and the plane $z = x + 2y$ and is bounded by the planes $x = 0$, $x = 2$, $y = 0$, and $y = 4$. Then find its volume.

CAS **33.** Use a computer algebra system to find the exact value of the integral $\iint_R x^5 y^3 e^{xy}\, dA$, where $R = [0, 1] \times [0, 1]$. Then use the CAS to draw the solid whose volume is given by the integral.

CAS **34.** Graph the solid that lies between the surfaces $z = e^{-x^2} \cos(x^2 + y^2)$ and $z = 2 - x^2 - y^2$ for $|x| \leqslant 1$, $|y| \leqslant 1$. Use a computer algebra system to approximate the volume of this solid correct to four decimal places.

35–36 Find the average value of f over the given rectangle.

35. $f(x, y) = x^2 y$, R has vertices $(-1, 0)$, $(-1, 5)$, $(1, 5)$, $(1, 0)$

36. $f(x, y) = e^y \sqrt{x + e^y}$, $R = [0, 4] \times [0, 1]$

37–38 Use symmetry to evaluate the double integral.

37. $\displaystyle\iint_R \frac{xy}{1 + x^4}\, dA$, $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$

38. $\displaystyle\iint_R (1 + x^2 \sin y + y^2 \sin x)\, dA$, $R = [-\pi, \pi] \times [-\pi, \pi]$

CAS **39.** Use your CAS to compute the iterated integrals

$$\int_0^1 \int_0^1 \frac{x - y}{(x + y)^3}\, dy\, dx \quad \text{and} \quad \int_0^1 \int_0^1 \frac{x - y}{(x + y)^3}\, dx\, dy$$

Do the answers contradict Fubini's Theorem? Explain what is happening.

40. (a) In what way are the theorems of Fubini and Clairaut similar?

(b) If $f(x, y)$ is continuous on $[a, b] \times [c, d]$ and

$$g(x, y) = \int_a^x \int_c^y f(s, t)\, dt\, ds$$

for $a < x < b$, $c < y < d$, show that $g_{xy} = g_{yx} = f(x, y)$.

15.3 Double Integrals over General Regions

For single integrals, the region over which we integrate is always an interval. But for double integrals, we want to be able to integrate a function f not just over rectangles but also over regions D of more general shape, such as the one illustrated in Figure 1. We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as in Figure 2. Then we define a new function F with domain R by

$$\boxed{1} \qquad F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

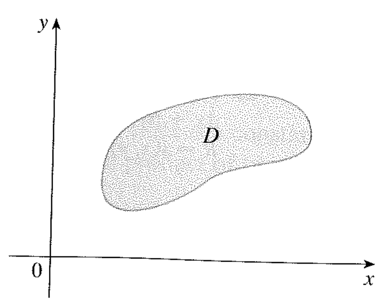

FIGURE 1

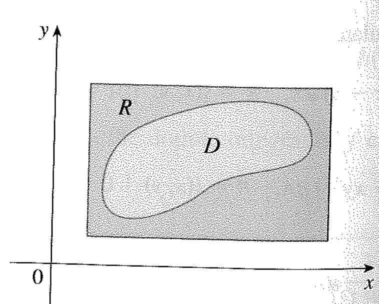

FIGURE 2