METU - NCC

	Calculus for Functions of Several Variables **Final Exam**	

Code : *Math 120* Acad.Year: *2011-2012* Semester : *Summer* Date : *10.8.2012* Time : *09:40* Duration : *180 minutes*	Last Name: Name : Student No.: Department: Section: Signature:
	9 QUESTIONS ON 8 PAGES TOTAL 100 POINTS

1 (8)	2 (12)	3 (12)	4 (8)	5 (11)	6 (14)	7 (20)	8 (10)	9 (5)		

1. (4+4 =8 pts) **a.** Write an equation of the line $\mathcal{L}$ that passes through (1,2,0) and perpendicular to $z = 0$ plane.

$z=0$ is the xy-plane whose normal is $\langle 0,0,1\rangle$. This vector will be a direction vector for our line.

Parametric equations for $\mathcal{L}$: $x=1$
$y=2$
$z=t$

b. Write an equation of the plane $\mathcal{P}$ which contains the line $\mathcal{L}$ in **(a)** and perpendicular to the plane $x + 2y - 3z = 4$

A normal vector for our line is $\vec{n} = \langle 0,0,1\rangle \times \langle 1,2,-3\rangle$

$$\begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 1 & 2 & -3 \end{vmatrix} = \langle -2,1,0\rangle. \quad P_0 = (1,2,0) \in \mathcal{P}$$

An equation for $\mathcal{P}$ is

$$-2(x-1) + y - 2 = 0 \Rightarrow -2x + y = 0$$

2. (6+6 =15 pts) This problem has two unrelated parts.

a. Find and classify the critical points of $f(x, y) = x^3 - 3xy + y^3$

$$\left.\begin{array}{l} f_x(x,y) = 3x^2 - 3y = 0 \Rightarrow y = x^2 \\ f_y(x,y) = -3x + 3y^2 = 0 \end{array}\right\} \Rightarrow -3x + 3x^4 = 0 \Rightarrow x(x^3-1) = 0$$

$x = 0$ or $x = 1$

$y = 0$ $\quad$ $y = 1$

$(0,0)$ & $(1,1)$ critical points

$f_{xx}(x,y) = 6x$, $f_{xy}(x,y) = -3$

$f_{yy}(x,y) = 6y$

$D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2$

$D(0,0) = -9 < 0 \Rightarrow (0,0)$ is a saddle point

$D(1,1) = 36 - 9 = 27 > 0$ & $f_{xx}(1,1) = 6 > 0 \Rightarrow (1,1)$ is a local minimum point

b. Find the absolute maximum and minimum values of $f(x, y) = x^2 + xy^2 + 4$ on $x^2 + y^2 = 1$

We use Lagrange multipliers with $g(x,y) = x^2 + y^2 = 1$.

$$\left.\begin{array}{l} \nabla f = \lambda \nabla g \\ g(x,y) = 1 \end{array}\right\} \begin{array}{l} \text{(i) } 2x + y^2 = 2\lambda x \\ \text{(ii) } 2xy = 2\lambda y \Rightarrow y(x-\lambda) = 0 \Rightarrow y = 0 \text{ or } x = \lambda \\ \text{(iii) } x^2 + y^2 = 1 \end{array}$$

If $y = 0$, (iii) $\Rightarrow x = \pm 1 \rightarrow (-1,0)$ and $(1,0)$

If $x = \lambda$, (i) $\Rightarrow y^2 = 2x^2 - 2x$, (iii) $\Rightarrow x^2 + 2x^2 - 2x - 1 = 0 \Rightarrow 3x^2 - 2x - 1 = 0$

$\Rightarrow (3x+1)(x-1) = 0 \Rightarrow x = -\frac{1}{3}$ or $x = 1$

$\Downarrow \qquad \Downarrow$

$y = \pm \frac{2\sqrt{2}}{3} \qquad y = 0$

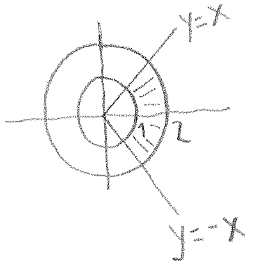
$\left.\begin{array}{l} f(-1,0) = 5 \\ f(1,0) = 5 \end{array}\right\} \rightarrow$ Absolute minimum value

$f\left(-\frac{1}{3}, \pm\frac{2\sqrt{2}}{3}\right) = \frac{1}{9} - \frac{1}{3}\cdot\frac{8}{9} + 4$

$= \frac{3 - 8 + 288}{27} = \frac{283}{27} \longrightarrow$ Absolute maximum value

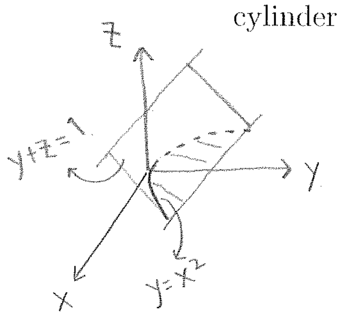
3. (4+4+4=12 pts) Compute the following multiple integrals.

a. $\iint_R e^{-(x^2+y^2)}dA$ where $R=\{(x,y,\cancel{z}) :\ |y|\le x \text{ and } 1\le x^2+y^2\le 4\}$



$$\int_{-\pi/4}^{\pi/4}\int_{1}^{2} e^{-r^2}\, r\, dr\, d\theta = \int_{-\pi/4}^{\pi/4}$$

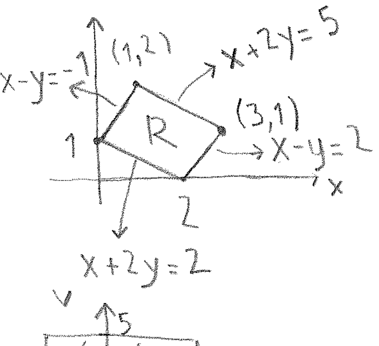
b. $\iiint_R dV$ where R is the region bounded by the planes $y+z=1$, $z=0$ and the cylinder $y=x^2$



$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\,dy\,dx = \int_{-1}^{1}\int_{x^2}^{1}(1-y)\,dy\,dx = \int_{-1}^{1}\left(y-\frac{y^2}{2}\right)\Big|_{x^2}^{1} dx$$

$$=\int_{-1}^{1}\left(\frac{1}{2}-x^2+\frac{x^4}{2}\right)dx = \left(\frac{1}{2}x-\frac{1}{3}x^3+\frac{1}{10}x^5\right)\Big|_{-1}^{1}=\frac{8}{15}$$

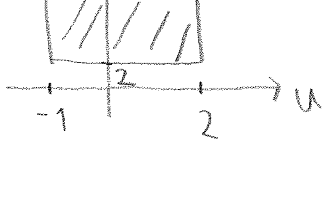
c. $\iint_R (y-x)dA$ where R is the parallelogram whose vertices are at $(0,1),(2,0),(1,2)$ and $(3,1)$. *(Hint : Use change of variables.)*



$u=x-y$

$v=x+2y$

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}1 & -1\\ 1 & 2\end{vmatrix}=3 \Rightarrow \frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{3}$$



$$\iint_R (y-x)\,dA=\int_{2}^{5}\int_{-1}^{2}(-u)\cdot\frac{1}{3}\,du\,dv=\frac{1}{3}\cdot 3\cdot\frac{-3}{2}=\frac{-3}{2}$$

6. (3+3+3+5=14 pts) Determine whether the following sequences are convergent or divergent. If convergent, then find the limit.

a. $a_n = (1+\frac{4}{n})^n$ $[1^\infty]$ Let $y = \left(1+\frac{4}{x}\right)^x \Rightarrow \ln y = x \ln\left(1+\frac{4}{x}\right)$

since $\ln$ is cont.

$$\Rightarrow \ln\left(\lim_{x\to\infty} y\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{4}{x}\right)}{1/x} \left[\frac{0}{0}\right]$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{-4}{x^2}\cdot\frac{x}{x+4}}{-1/x^2} = \lim_{x\to\infty} \frac{4x^3}{x^2(x+4)} = 4 \Rightarrow \lim_{x\to\infty} y = e^4 = \lim_{n\to\infty} e^4$$

b. $b_n = \cos(n\pi)\frac{n+1}{n^2} = (-1)^n \frac{n+1}{n^2}$

$$-\frac{n+1}{n^2} \leqslant b_n \leqslant \frac{n+1}{n^2}$$

$-\frac{n+1}{n^2} \xrightarrow{n\to\infty} 0$, $\frac{n+1}{n^2} \xrightarrow{n\to\infty} 0$

by squeeze theorem $\lim_{n\to\infty} b_n = 0$

c. $c_n = \dfrac{(n!)^2}{(2n)!}$

Consider $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ and apply ratio test $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$$= \lim_{n\to\infty} \frac{((n+1)!)^2}{(2(n+1))!} \frac{(2n)!}{(n!)^2} = \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$

series converges $\Rightarrow \lim_{n\to\infty} c_n = 0.$

d. Let $d_1 = 2$ and $d_{n+1} = \dfrac{1}{3-d_n}$ (*Hint: d_n is decreasing implies $d_n > 0$*)

First, show that d_n is decreasing by induction: For $d_2 = 1 < d_1 = 2$.

Assuming $d_{k+1} < d_k$, $d_{k+2} = \frac{1}{3-d_{k+1}} < \frac{1}{3-d_k} = d_{k+1}$. So d_n is decreasing

since $3-d_{k+1} > 3-d_k$

d_n decreasing $\Rightarrow \frac{1}{3-d_n} > 0$ for all n since $d_n \leqslant 2$.

So $\{d_n\}$ is decreasing and bounded below. Hence it must be convergent, i.e., $\lim_{n\to\infty} d_n = L$ exists $\Rightarrow L = \frac{1}{3-L} \Rightarrow L^2 - 3L + 1 = 0$

$\cancel{L = \frac{3+\sqrt{9-4}}{2}}$ (since $d_n \leqslant 2$)

OR

$$\boxed{L = \frac{3-\sqrt{5}}{2}}$$

4. (5+3=8 pts) Let F be the vector field $F(x,y) = \overbrace{(3x^2+y^2)}^{P}\mathbf{i} + \overbrace{(3y^2+2xy)}^{Q}\mathbf{j}$

a. Determine whether F is conservative or not. If it is conservative, find a potential function for F.

$\frac{\partial Q}{\partial x} = 2y = \frac{\partial P}{\partial y}$ & Domain of F is $\mathbb{R}^2$ which is simply connected

$\Rightarrow F$ is conservative, i.e., $F = \nabla f = \langle f_x, f_y \rangle$

$f_x = 3x^2+y^2 \Rightarrow f(x,y) = x^3 + xy^2 + h(y)$

$f_y = 2xy + h'(y) = 3y^2 + 2xy \Rightarrow h(y) = y^3 + K$ where K is a constant

$f(x,y) = x^3 + xy^2 + y^3 + K$ is a potential function for F.

b. Evaluate $\int_C F \cdot dr$ where C is the part of the curve $y = (x-1)^2(x+1)^2$ from the point $(-1,0)$ to $(1,0)$. fundamental thm. for line integrals

$\int_C F \cdot dr = \int_C \nabla f \cdot dr \stackrel{\vee}{=} f(1,0) - f(-1,0) = 1 - (-1) = 2$

5. (3+3+5=11 pts) Let F be the vector field $F(x,y) = \overset{P}{\dfrac{x}{x^2+y^2}}\mathbf{i} + \overset{Q}{\dfrac{y}{x^2+y^2}}\mathbf{j}$

a. Compute $\int_C F \cdot dr$ on the circle $x^2+y^2=1$ oriented counterclockwise.

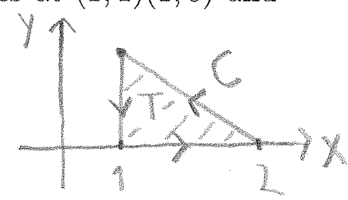
Notice that F is not defined at the origin

Parametrization for C: $x = \cos\theta$, $y = \sin\theta$ $\quad 0 \leq \theta \leq 2\pi$

$$\int_C F.dr = \int_0^{2\pi} F(r(\theta)) \cdot r'(\theta)\, d\theta = \int_0^{2\pi} \langle \cos\theta, \sin\theta \rangle \cdot \langle -\sin\theta, \cos\theta \rangle d\theta$$

$$= 0$$

b. Compute $\int_C F \cdot dr$ on the boundary of the triangle with vertices at $(1,1)(1,0)$ and $(2,0)$ oriented counterclockwise.



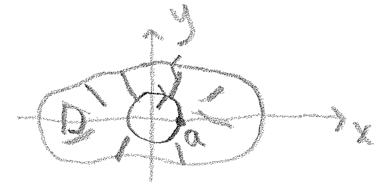
Since T is simply-connected over which F is defined and $\dfrac{\partial Q}{\partial x} = \dfrac{-2xy}{(x^2+y^2)^2} = \dfrac{\partial P}{\partial y}$

F is conservative $\Rightarrow \int_C F.dr = 0$ since C is a closed curve.

c. Show that $\int_C F \cdot dr = 0$ on any closed curve C. (*Hint : Use Green's theorem.*)

If C is a closed curve not around the origin, then we know part (b) that F is conservative over the region bounded by C $\Rightarrow \int_C F.dr = 0$.

If C is a curve around the origin,



Apply Green's thm on the shaded region D with suitable orientations

$$\int_{C \cup C'} F.dr = 0 \Rightarrow \int_C F.dr = \int_{C'} F.dr = 0$$

7. (4x5 =20 pts) Determine whether the following series converge or diverge. Explain the tests used and give all necessary details.

a. $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ Integral test

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{t\to\infty} \int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = -\frac{1}{2}\lim_{t\to\infty}\int_{-1}^{-\sqrt{t}} e^u\,du = \frac{1}{2}\lim_{t\to\infty}\left(\frac{1}{e^{\sqrt{t}}} - \frac{1}{e}\right)$$

$u = -\sqrt{x}$

$du = \frac{-1}{2\sqrt{x}}\,dx$

$= -\frac{1}{2e}$ convergent

$\Rightarrow$ Series converges.

b. $\sum_{n=2}^{\infty} \sin\left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \sum_{n=2}^{\infty} \sin\left(\frac{2}{n^2-1}\right)$ limit comparison test with $\sum \frac{2}{n^2-1}$ which is convergent

$$\lim_{n\to\infty} \frac{\sin\left(\frac{2}{n^2-1}\right)}{\frac{2}{n^2-1}} = 1$$

Hence the series converges.

c. $\sum_{n=1}^{\infty} \frac{2n+1}{n\sqrt{n}+n^2\sqrt{n+1}}$ limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is convergent by p-test $\left(\frac{3}{2}>1\right)$

$$\lim_{n\to\infty} \frac{\frac{2n+1}{n\sqrt{n}+n^2\sqrt{n+1}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^{5/2}\left(2+1/n^{5/2}\right)}{n^{5/2}\left(\frac{1}{n}+\sqrt{1+\frac{1}{n}}\right)} = 2.$$

Hence the series converges.

d. $\sum_{n=1}^{\infty} (-1)^{n+1} n^2 e^{-n^2}$ alternating series

(i) $\lim_{n\to\infty} \frac{n^2}{e^{n^2}} = 0$

(ii) Let $f(x) = \frac{x^2}{e^{x^2}}$. $f'(x) = \frac{2x(1-x^2)}{e^{x^2}} \leqslant 0$ for $x \geqslant 1 \Rightarrow \left\{\frac{n^2}{e^{n^2}}\right\}$ decreasing

Hence the series converges by AST

e. $\sum_{n=2}^{\infty} \frac{1}{\ln(n \ln n)}$ Recall $\ln n < n$ and $\ln(x)$ is an increasing functi

$$\frac{1}{2}\cdot\frac{1}{n} < \frac{1}{2\ln(n)} = \frac{1}{\ln(n^2)} < \frac{1}{\ln(n\ln n)}$$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent

$\sum_{n=2}^{\infty} \frac{1}{\ln(n\ln n)}$ divergent by comparison test.

8. (3+2+5=10 pts) Given $f(x) = \dfrac{x^2}{16 + x^4}$

a. Find the power series of $f(x)$.

$$\frac{x^2}{16\left(1-\left(\frac{-x^4}{16}\right)\right)} = \frac{x^2}{16}\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n}}{16^n} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{16^{n+1}}$$

b. Compute the radius of convergence of the power series found in **(a)**.

Interval of convergence: $-1 < -\frac{x^4}{16} < 1 \Rightarrow -16 < x^4 < 16$
$\Rightarrow -2 < x < 2$

Radius of convergence $= 2$

c. Estimate $\displaystyle\int_0^1 \frac{x^2}{16 + x^4}dx$ with an error less than 0.0001.

$$\int \frac{x^2}{16+x^4}dx = \int \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{16^{n+1}}dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{(4n+3)16^{n+1}}$$

$$\int_0^1 \frac{x^2}{16+x^4}dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+3)16^{n+1}} = \frac{1}{3.16} - \frac{1}{7.16^2} - \frac{1}{11.16^3}$$

$$\int_0^1 \frac{x^2}{16+x^4}dx \approx \frac{1}{3.16} - \frac{1}{7.16^2} \qquad |\text{error}| < \frac{1}{11.16^3} < \frac{1}{10000}$$

9. (5 pts) Write the Taylor series of $f(x) = \sin(x)$ at the center $\frac{\pi}{2}$

$f^{(0)}(x) = f(x) = \sin x \qquad f\left(\frac{\pi}{2}\right) = 1$

$f'(x) = \cos x \qquad f'\left(\frac{\pi}{2}\right) = 0$

$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{2}\right) = -1$

$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{2}\right) = 0$

$$\sin x = \sum_{n=0}^{\infty}\frac{f^{(n)}\left(\frac{\pi}{2}\right)}{n!}\left(x-\frac{\pi}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{2}\right)^{2n}$$