METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES MIDTERM 1	
Code : *MAT 120* Acad.Year: *2012-2013* Semester : *FALL* Date : *17.11.2012* Time : *14:40* Duration : *110 minutes*	Last Name: Name : Solutions Department: Signature: Student No.: Section:
	7 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1. (6)	2. (14)	3. (8)	4. (12)	5. (16)	6. (14)	7. (20)	8. (10)	Bonus	

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

Good luck!

1.*(2× 3pts)* The yz-plane is the plane in 3D containing the y and z axes.

(A) Write an equation for the yz-plane.

$$\boxed{x=0}$$

(B) Write a vector parallel to the yz-plane. Show it is parallel.

$x=0$ has normal vector $\bar{n} = \langle 1, 0, 0 \rangle$

All vectors of the form $\boxed{\bar{v} = \langle 0, a, b \rangle}$

will be parallel to $x=0$ b/c $\bar{n} \cdot \bar{v} = 0$.

2.*(2× 7pts)* Give an equation for the tangent plane to the following surfaces at the indicated points.

(A) The surface $z = x^y$ at the point $p = (2, 3)$.

$f(x,y) = x^y$ | $f(2,3) = 2^3 = 8$

$f_x(x,y) = y x^{y-1}$ | $f_x(2,3) = 3 \cdot 2^2 = 12$

$f_y(x,y) = x^y \ln x$ | $f_y(2,3) = 2^3 \ln 2 = 8 \ln 2$

Tangent plane: $\boxed{z = 12(x-2) + 8\ln 2 (y-3) + 8}$

(B) The surface $\sin(xy) + \cos(yz) = 1$ at the point $p = (\frac{1}{6}, \pi, \frac{1}{3})$.

$F(x,y,z) = \sin(xy) + \cos(yz)$

$F_x(x,y,z) = y\cos(xy)$ | $F_x(\frac{1}{6}, \pi, \frac{1}{3}) = \pi \cdot \frac{\sqrt{3}}{2}$

$F_y(x,y,z) = x\cos(xy) - z\sin(yz)$ | $F_y(\frac{1}{6}, \pi, \frac{1}{3}) = \frac{1}{6}\frac{\sqrt{3}}{2} - \frac{1}{3}\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{12}$

$F_z(x,y,z) = -y\sin(yz)$ | $F_z(\frac{1}{6}, \pi, \frac{1}{3}) = -\pi\frac{\sqrt{3}}{2}$

Tangent plane: $\boxed{\frac{\pi\sqrt{3}}{2}(x - \frac{1}{6}) - \frac{\sqrt{3}}{12}(y-\pi) - \frac{\pi\sqrt{3}}{2}(z - \frac{1}{3}) = 0}$

3. *(8pts)* Write the equation for a hyperbolic paraboloid whose x-traces are hyperbolas and whose y and z-traces are parabolas.

$$\boxed{\begin{array}{c} x = y^2 - z^2 + c \\ \text{or} \\ x = z^2 - y^2 + c \end{array}}$$

EX: $x = y^2 - z^2$

x-traces: $c = y^2 - z^2$ (hyperbola)

y-traces: $x = c^2 - z^2$ (parabola)

z-traces: $x = y^2 - c^2$ (parabola)

4. *(12pts)* Consider a point moving along the curve $\mathbf{r}(t)$. The point's accelleration $\mathbf{r}''(t)$ can be written as a sum of two parts $\mathbf{r}'' = \mathbf{a}_T + \mathbf{a}_N$.

- The tangent accelleration $\mathbf{a}_T$ is tangent to the curve – it is due to the point speeding up.
- The normal accelleration $\mathbf{a}_N$ is perpendicular to the curve – it is due to the curve turning.

Find formulas for $\mathbf{a}_T$ and $\mathbf{a}_N$ if $\mathbf{r}(t) = \langle t,\ t^2,\ t^3\rangle$

$\bar{r}(t) = \langle t, t^2, t^3\rangle$ (position)

$\bar{r}'(t) = \langle 1, 2t, 3t^2\rangle$ (velocity)

$\bar{r}''(t) = \langle 0, 2, 6t\rangle$ (accelleration)

Want: $\bar{r}''(t) = \bar{a}_T(t) + \bar{a}_N(t)$

$\left\{\begin{array}{l} \bar{a}_T(t) \text{ tangent to curve} \\ \bar{a}_N(t) \perp \text{ to curve} \\ (\text{Recall: } \bar{r}' \text{ is tangent to curve}) \end{array}\right.$

$$\bar{a}_T(t) = \text{Proj}_{\bar{r}'}\, \bar{r}''$$

$$= \frac{\bar{r}' \cdot \bar{r}''}{r' \cdot r'}\, r' = \frac{\langle 1, 2t, 3t^2\rangle \cdot \langle 0, 2, 6t\rangle}{\langle 1, 2t, 3t^2\rangle \cdot \langle 1, 2t, 3t^2\rangle} \langle 1, 2t, 3t^2\rangle$$

$$\boxed{\bar{a}_T(t) = \frac{4t + 18t^3}{1 + 4t^2 + 9t^4} \langle 1, 2t, 3t^2\rangle}$$

$$\bar{a}_N(t) = \text{Proj}^{\perp}_{\bar{r}'}\, \bar{r}''$$

$$= \bar{r}'' - \text{Proj}_{\bar{r}'}\, \bar{r}'' = \langle 0, 2, 6t\rangle - \frac{4t + 18t^3}{1 + 4t^2 + 9t^4} \langle 1, 2t, 3t^2\rangle$$

$$\boxed{\bar{a}_N(t) = \left\langle -\frac{4t + 18t^3}{1 + 4t^2 + 9t^4},\ 2 - \frac{8t^2 + 36t^4}{1 + 4t + 9t^4},\ 6t - \frac{12t^3 + 54t^5}{1 + 4t + 9t^4} \right\rangle}$$

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5.*(4×4pts)* Compute the following limits, or show that they don't exist.

(A) $\lim_{(x,y)\to(0,0)} \frac{2xy+1}{x^2+y^2-1}$

this function is defined at (0,0)
→ so it is continuous at (0,0)

$$\lim_{(x,y)\to(0,0)} \frac{2xy+1}{x^2+y^2-1} = \frac{1}{-1} = \boxed{-1}$$

(B) $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$

path: $\left.\begin{matrix} x=0 \\ y\to 0\end{matrix}\right\}$ $\lim_{y\to 0} \frac{0}{y^2} = 0$

$\ast$

path: $\left.\begin{matrix} x=y \\ y\to 0\end{matrix}\right\}$ $\lim_{y\to 0} \frac{2y^2}{y^2+y^2} = 1$

Different paths give different limits. So this limit $\boxed{\text{D.N.E.}}$

(C) $\lim_{(x,y)\to(0,0)} \frac{x^3y^4}{3x^9+2y^6}$

path: $\left.\begin{matrix} y=0 \\ x\to 0\end{matrix}\right\}$ $\lim_{x\to 0} \frac{0}{3x^9} = 0$

$\ast$

path: $\left.\begin{matrix} x^3=y^2 \\ x\to 0\end{matrix}\right\}$ $\lim_{x\to 0} \frac{x^3 \cdot x^6}{3x^9+2x^9} = 1/5$

Different paths give different limits. So this limit $\boxed{\text{D.N.E.}}$

(D) $\lim_{(x,y)\to(1,1)} \frac{x}{x+y}$

this function is defined at (1,1)
→ so it is continuous at (1,1)

$$\lim_{(x,y)\to(1,1)} \frac{x}{x+y} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

6. *(2× 7pts)* The following parts are about calculating partial derivatives.

(A) Calculate $\frac{\partial x}{\partial z}$ if x, y, z satisfy the equation

$$e^{xy} - \cos(xz) = 0. \leftarrow F(x,y,z) = 0$$

$$F(x,y,z) = e^{xy} - \cos(xz)$$

$$\boxed{\frac{\partial x}{\partial z} = -\frac{\partial F/\partial z}{\partial F/\partial x} = -\frac{x\sin(xz)}{ye^{xy} + z\sin(xz)}}$$

(B) Calculate $\frac{\partial^2 y}{\partial x \partial z}$ if x, y, z satisfy the equation

$$x^3y + y^3z + z^3x = 1. \leftarrow F(x,y,z) = 1$$

$$\frac{\partial^2 y}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial z}\right)$$

$$\frac{\partial y}{\partial z} = -\frac{\partial F/\partial z}{\partial F/\partial y} = -\frac{3z^2x + y^3}{3y^2z + x^3}$$

$$\frac{\partial^2 y}{\partial x \partial z} = \frac{\partial}{\partial x}\left(-\frac{3z^2x + y^3}{3y^2z + x^3}\right)$$

$$= -\frac{(3z^2)\cdot(3y^2z + x^3) - (3z^2x + y^3)(3x^2)}{(3y^2z + x^3)^2}$$

$$= \frac{9x^3z^2 + 3x^2y^3 - 9y^2z^3 - 3x^3z^2}{(3y^2z + x^3)^2}$$

$$= \boxed{\frac{6x^3z^2 + 3x^2y^3 - 9y^2z^3}{(3y^2z + x^3)^2}}$$

7. *(8+12pts)* The following parts are about extremal values (max/min).

(A) Find the extremal values of the function $f(x,y) = -3x + 2y$ subject to the condition $x^2 + \frac{y^2}{4} = 1$.

Lagrange Multipliers

$f(x,y) = -3x + 2y \rightarrow \nabla f = \langle -3, 2 \rangle$

$g(x,y) = x^2 + y^2/4 \rightarrow \nabla g = \langle 2x, y/2 \rangle$

Solve: $\begin{cases} -3\lambda = 2x \\ 2\lambda = y/2 \\ x^2 + y^2/4 = 1 \end{cases}$ $\leftrightarrow \begin{cases} 6\lambda = -4x \\ 6\lambda = 3y/2 \end{cases}$ So $\boxed{y = -\frac{8}{3}x}$

plug in

$x^2 + \frac{16}{9}x^2 = 1$

$\frac{25}{9}x^2 = 1$

$x = \pm 3/5$

if $x = 3/5$, $y = -8/5$ and $f = -9/5 - 16/5 = -5$ MIN

if $x = -3/5$, $y = 8/5$ and $f = 9/5 + 16/5 = 5$ MAX

(B) Find the extremal values of the function $k(x,y) = -3x + 2y + 1 - x^2 - \frac{y^2}{4}$ over the domain $x^2 + \frac{y^2}{4} \leqslant 1$.

Inside Domain

critical points of k:

$\begin{cases} 0 = k_x = -3 - 2x \\ 0 = k_y = 2 - y/2 \end{cases}$

So $\begin{cases} x = -3/2 \\ y = 4 \end{cases}$

Note: $(-3/2, 4)$ is not in $x^2 + y^2/4 \leq 1$

$\Rightarrow$ All extremal values are on boundary!

② On Boundary

Instead of solving this problem from scratch, note that on the boundary of $x^2 + y^2/4 \leq 1$ we have $x^2 + y^2/4 = 1$

So here k is

$k = -3x + 2y + 1 - 1$
$= -3x + 2y$

This problem was solved in (A).

- min at $(3/5, -8/5) \rightarrow k = -5$
- max at $(-3/5, 8/5) \rightarrow k = 5$

8. *(10pts)* At the point $(1,-1)$ the function $f(x,y) = ax^2 + bxy + cy^2$ has directional derivative in the direction $\mathbf{v} = \langle 2,1\rangle$ equal to $D_{\mathbf{v}} f(1,-1) = 0$. Compute $f(1,-1)$.

$$f(x,y) = ax^2 + bxy + cy^2$$
$$\nabla f = \langle 2ax + by,\ bx + 2cy\rangle$$
$$\nabla f(1,-1) = \langle 2a-b,\ b-2c\rangle$$

$$0 = D_{\vec v} f(1,-1) = |\mathrm{Proj}_{\vec v} \nabla f| = \frac{\langle 2,1\rangle \cdot \langle 2a-b,\ b-2c\rangle}{\sqrt{5}}$$
$$0 = 4a - 2b + b - 2c$$
$$0 = 4a - b - 2c \rightarrow \underline{b = 4a - 2c}$$
$$f(1,-1) = a - b + c$$
$$= a - (4a - 2c) + c$$
$$= \boxed{-3a + 3c}$$

Other correct answers:

$3a - \frac{3}{2} b$

or

$-\frac{3}{4} b + \frac{3}{2} c$

BONUS

Suppose $z = f(x,y)$ and $x = x(s,t)$, $y = y(s,t)$. Write a formula for the mixed partial $\dfrac{\partial^2 z}{\partial s\, \partial t}$.

$$\frac{\partial^2 z}{\partial s \partial t} = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial t}\right) = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\right)$$

$$= \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\cdot\frac{\partial}{\partial s}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial}{\partial s}\left(\frac{\partial y}{\partial t}\right)$$

$$= \left(\frac{\partial \frac{\partial z}{\partial x}}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial \frac{\partial z}{\partial x}}{\partial y}\cdot\frac{\partial y}{\partial s}\right)\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\,\frac{\partial \frac{\partial x}{\partial t}}{\partial s}$$
$$+ \left(\frac{\partial \frac{\partial z}{\partial y}}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial \frac{\partial z}{\partial y}}{\partial y}\cdot\frac{\partial y}{\partial s}\right)\cdot\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\,\frac{\partial \frac{\partial y}{\partial t}}{\partial s}$$

$$= \boxed{\begin{aligned}&\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial x}{\partial s}\cdot\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial y}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s \partial t}\\ &+ \frac{\partial^2 z}{\partial x \partial y}\cdot\frac{\partial x}{\partial s}\cdot\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s}\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s \partial t}\end{aligned}}$$

I have always found these two terms suprising