

**M E T U - N C C**  
**Mathematics Group**

Calculus with Analytic Geometry					
Midterm Exam					
Code : MAT 120			Last Name :		
Acad. Year : 2010-2011			Name :		Stud. No :
Semester : Summer			Dept. :		Sec. No :
Instructors : A.D./E.G./O.K.			Signature :		
Date : 25.07.2011			6 Questions on 8 Pages Total 100 Points		
Time : 17.40					
Duration : 120 minutes					
1 (12)	2 (15)	3 (20)	4 (15)	5 (14)	6 (24)

**Q.1 (6 + 6 = 12 pts)** Evaluate the following limits:

(a)  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4}$

Path 1:  $x = t^2, y = t$       $\lim_{t \rightarrow 0} \frac{t^4}{t^4 + t^4} = \frac{1}{2}$

Path 2:  $x = t, y = 0$       $\lim_{t \rightarrow 0} \frac{0}{t^2 + 0} = 0$  (not equal)

So the limit does not exist.

(b)  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 \sin^2(5y)}{x^2 y^4 + y^2}$

$$\frac{x^2 \sin^2(5y)}{x^2 y^4 + y^2} = \left( \frac{x^2}{x^2 y^2 + 1} \right) \left( \frac{\sin^2(5y)}{y^2} \right)$$

$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 y^2 + 1} = 0$  (a rational function, numerator goes to 0, denominator doesn't)

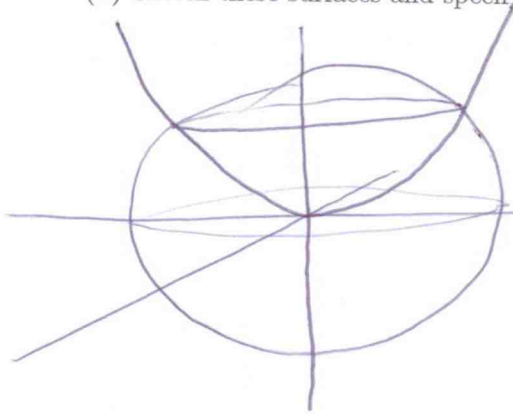
$\lim_{(x,y) \rightarrow (0,0)} \left( \frac{\sin^2(5y)}{y^2} \right) = 25$  from the one variable limit.

Both limits exist, so we can multiply.

$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 \sin^2(5y)}{x^2 y^4 + y^2} = \boxed{0}$

**Q.2 (5 × 3 = 15 pts)** Consider the following quadric surfaces  $x^2 + y^2 + z^2 = 8$  and  $x^2 + y^2 = 2z$ .

(a) Sketch these surfaces and specify their types.



$$x^2 + y^2 + z^2 = 8 = \text{sphere}$$

$$x^2 + y^2 = 2z = \text{elliptic paraboloid}$$

(b) Find the angle between their tangent planes at  $(1, \sqrt{3}, 2)$  (Leave your answer in the form  $\arccos(\theta)$ ).

Find normals:  $\vec{n}_1 = \nabla(x^2 + y^2 + z^2) \Big|_{(1, \sqrt{3}, 2)} = (2, 2\sqrt{3}, 4)$

$$\vec{n}_2 = \nabla(x^2 + y^2 - 2z) \Big|_{(1, \sqrt{3}, 2)} = (2, 2\sqrt{3}, -2)$$

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} = \frac{4 + 12 - 8}{\sqrt{4 + 12 + 16} \sqrt{4 + 12 + 4}} = \frac{8}{\sqrt{32} \sqrt{20}} = \frac{1}{\sqrt{10}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{10}}\right)$$

(c) Find the parametric equations of the tangent line to their intersection curve at  $(1, \sqrt{3}, 2)$ .

We can find one tangent vector as  $\vec{n}_1 \times \vec{n}_2$

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2\sqrt{3} & 4 \\ 2 & 2\sqrt{3} & -2 \end{vmatrix} = -10\sqrt{3}\hat{i} + 12\hat{j} = \langle -10\sqrt{3}, 12, 0 \rangle$$

Parametric eqns:

$$(1, \sqrt{3}, 2) + t(-10\sqrt{3}, 12, 0) = \left( 1 - 10\sqrt{3}t, \sqrt{3} + 12t, 2 \right) \quad t \in \mathbb{R}$$

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Q.3 (20 pts) Find the extreme values of the function  $f(x, y) = 2x^2 + 3y^2 - 4x - 5$  on the region described by the inequality  $x^2 + y^2 \leq 16$  (Use Lagrange multipliers for the boundary).

Interior points:  $\nabla f = \vec{0}$

$$(4x - 4, 6y) = (0, 0) \Leftrightarrow \boxed{x=1, y=0} \quad \boxed{(1, 0)}$$

(inside the region)

Boundary; let  $g(x, y) = x^2 + y^2$ .

We are looking for the points where

$$\nabla f = \lambda \nabla g \text{ for some } \lambda.$$

$$(4x - 4, 6y) = \lambda (2x, 2y)$$

$$4x - 4 = \lambda \cdot 2x$$

$$6y = \lambda \cdot 2y$$

$$x^2 + y^2 = 16$$

$$\left. \begin{array}{l} 4x - 4 = \lambda \cdot 2x \\ 6y = \lambda \cdot 2y \end{array} \right\} \begin{array}{l} \rightarrow y = 0 \text{ or } \lambda = 3 \\ \downarrow \\ x = \pm 4 \end{array}$$

$$\boxed{(4, 0) \text{ and } (-4, 0)}$$

$$\begin{array}{l} \lambda = 3 \\ \downarrow \\ 2x = -4 \\ x = -2 \end{array}$$

$$y = \pm 2\sqrt{3}$$

$$\boxed{(-2, 2\sqrt{3}) \text{ and } (-2, -2\sqrt{3})}$$

We have 5 points to check:  $(1, 0)$ ,  $(4, 0)$ ,  $(-4, 0)$ ,  $(-2, 2\sqrt{3})$  and  $(-2, -2\sqrt{3})$

$$f(1, 0) = 2 - 4 - 5 = \boxed{-7}$$

$$f(4, 0) = 32 - 16 - 5 = 11$$

$$f(-4, 0) = 32 + 16 - 5 = 43$$

$$f(-2, 2\sqrt{3}) = 8 + 36 + 8 - 5 = \boxed{47}$$

$$f(-2, -2\sqrt{3}) = 8 + 36 + 8 - 5 = \boxed{47}$$

maximum value:  $\boxed{47}$   
(at  $(-2, 2\sqrt{3})$  and  $(-2, -2\sqrt{3})$ )

minimum value:  $\boxed{-7}$   
(at  $(1, 0)$ )

Q.4 (5×3=15 pts) Consider the surface  $x^2y + z^3x + y^2z = 8$ .

(a) Find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  at  $(1, 0, 2)$  by implicit differentiation.

$$h(x, y, z) = x^2y + z^3x + y^2z$$

$$\frac{\partial z}{\partial x} = \frac{-\partial h / \partial x}{\partial h / \partial z} = \frac{-(2xy + z^3)}{3z^2x + y^2}, \quad \left. \frac{\partial z}{\partial x} \right|_{(1, 0, 2)} = \frac{-8}{12} = \boxed{\frac{-2}{3}}$$

$$\frac{\partial z}{\partial y} = \frac{-\partial h / \partial y}{\partial h / \partial z} = \frac{-(x^2 + 2yz)}{3z^2x + y^2}, \quad \left. \frac{\partial z}{\partial y} \right|_{(1, 0, 2)} = \boxed{\frac{-1}{12}}$$

(b) Find the tangent plane equation to the surface at  $(1, 0, 2)$ .

$$z - 2 = \left. \frac{\partial z}{\partial x} \right|_{(1, 0, 2)} (x - 1) + \left. \frac{\partial z}{\partial y} \right|_{(1, 0, 2)} (y - 0)$$

$$\boxed{z - 2 = \frac{-2}{3}(x - 1) - \frac{1}{12}y}$$

(c) Find an approximate solution for  $z$  to the equation  $(0.9)^2(-0.1) + z^3(0.9) + (-0.1)^2z = 8$  near  $z = 2$  (Use the linear approximation).

$$(x_0, y_0, z_0) = (1, 0, 2)$$

$$dx = \Delta x = x - x_0 = 0.9 - 1 = -0.1$$

$$dy = \Delta y = y - y_0 = \cancel{(-0.1)} - 0 = -0.1$$

$$\Delta z \approx dz = \left. \frac{\partial z}{\partial x} \right|_{(1, 0, 2)} dx + \left. \frac{\partial z}{\partial y} \right|_{(1, 0, 2)} dy$$

$$= \frac{-2}{3}(-0.1) - \frac{1}{12}(-0.1) = +0.075$$

$$z = z_0 + \Delta z \approx z_0 + dz = 2 + 0.075$$

$$\boxed{= 2.075}$$

