METU – NCC
Mathematics Group

	Calculus with Analytic Geometry Midterm Exam	
Code : MAT 120 Acad.Year : 2010-2011 Semester : Summer Instructors : A.D./E.G./O.K. Date : 25.07.2011 Time : 17.40 Duration : 120 minutes	Last Name : Name : Stud. No : Dept. : Sec. No : Signature : **6** Questions on **8** Pages Total **100** Points	

1 (12)	2 (15)	3 (20)	4 (15)	5 (14)	6 (24)		

Q.1 (6 +6 = 12 pts) Evaluate the following limits:

(a) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

Path 1: $x = t^2$, $y = t$ $\quad \lim_{t\to 0} \frac{t^4}{t^4+t^4} = \frac{1}{2}$

Path 2: $x = t$, $y = 0$ $\quad \lim_{t\to 0} \frac{0}{t^2+0} = 0$

not equal

So the limit does not exist.

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2(5y)}{x^2y^4+y^2}$

$$\frac{x^2\sin^2(5y)}{x^2y^4+y^2} = \left(\frac{x^2}{x^2y^2+1}\right)\left(\frac{\sin^2(5y)}{y^2}\right)$$

$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2y^2+1} = 0$ (a rational function, numerator goes to 0, denominator doesn't)

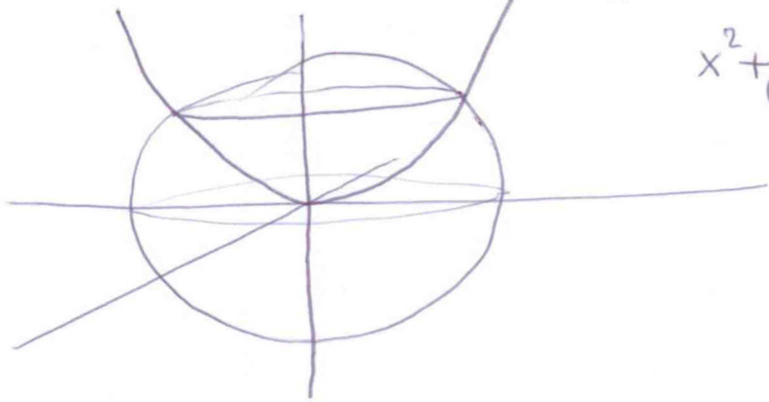
$\lim_{(x,y)\to(0,0)} \left(\frac{\sin^2(5y)}{y^2}\right) = 25$ from the one variable limit.

Both limits exist, so we can multiply.

$$\lim \frac{x^2\sin^2(5y)}{\ldots} = \boxed{0}$$

Q.2 (5 × 3 = 15 pts) Consider the following quadric surfaces $x^2 + y^2 + z^2 = 8$ and $x^2 + y^2 = 2z$.

(a) Sketch these surfaces and specify their types.



$x^2+y^2+z^2=8$: sphere

$x^2+y^2=2z$: elliptic paraboloid

(b) Find the angle between their tangent planes at $(1, \sqrt{3}, 2)$ (Leave your answer in the form $\arccos(\theta)$).

Find normals: $\vec{n}_1 = \nabla(x^2+y^2+z^2)\Big|_{(1,\sqrt{3},2)} = (2, 2\sqrt{3}, 4)$

$\vec{n}_2 = \nabla(x^2+y^2-2z)\Big|_{(1,\sqrt{3},2)} = (2, 2\sqrt{3}, -2)$

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{4+12-8}{\sqrt{4+12+16}\,\sqrt{4+12+4}} = \frac{8}{\sqrt{32}\,\sqrt{20}} = \frac{1}{\sqrt{10}}$$

$$\boxed{\theta = \arccos\left(\frac{1}{\sqrt{10}}\right)}$$

(c) Find the parametric equations of the tangent line to their intersection curve at $(1, \sqrt{3}, 2)$.

We can find one tangent vector as $\vec{n}_1 \times \vec{n}_2$

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 2 & 2\sqrt{3} & 4 \\ 2 & 2\sqrt{3} & -2 \end{vmatrix} = -10\sqrt{3}\, i + 12 j = \langle -10\sqrt{3}, 12, 0 \rangle$$

Parametric eqns:

$$(1,\sqrt{3},2) + t(-10\sqrt{3}, 12, 0) = \boxed{\begin{array}{c}(1-10\sqrt{3}t, \sqrt{3}+12t, 2) \\ t \in \mathbb{R}\end{array}}$$

Last Name: **Name:**

Q.3 (20 pts) Find the extreme values of the function $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ on the region described by the inequality $x^2 + y^2 \leq 16$ (Use Lagrange multipliers for the boundary).

Interior points: $\nabla f = \vec{0}$

$(4x-4, 6y) = (0,0) \Leftrightarrow \boxed{x=1,\ y=0}$. $\boxed{(1,0)}$

(inside the region)

Boundary; let $g(x,y) = x^2+y^2$.

We are looking for the points where

$\nabla f = \lambda \nabla g$ for some λ.

$(4x-4, 6y) = \lambda(2x, 2y)$

$4x-4 = \lambda \cdot 2x$

$6y = \lambda \cdot 2y$

$x^2+y^2 = 16$

$\rightarrow y=0$ or $\lambda = 3$

$y = 0 \rightarrow x = \mp 4$ $\boxed{(4,0) \text{ and } (-4,0)}$

$\lambda = 3 \rightarrow 2x = -4$, $x = -2$, $y = \mp 2\sqrt{3}$ $\boxed{(-2, 2\sqrt{3}) \text{ and } (-2, -2\sqrt{3})}$

We have 5 points to check: $(1,0)$, $(4,0)$, $(-4,0)$, $(-2, 2\sqrt{3})$ and $(-2,-2\sqrt{3})$

$f(1,0) = 2-4-5 = -7$

$f(4,0) = 32-16-5 = 11$

$f(-4,0) = 32+16-5 = 43$

$f(-2, 2\sqrt{3}) = 8+36+8-5 = 47$

$f(-2,-2\sqrt{3}) = 8+36+8-5 = 47$

maximum value: $\boxed{47}$ (at $(-2, 2\sqrt{3})$ and $(-2, -2\sqrt{3})$)

minimum value: $\boxed{-7}$ (at $(1,0)$).

Q.4 (5×3=15 pts) Consider the surface $x^2y + z^3x + y^2z = 8$.

(a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(1, 0, 2)$ by implicit differentiation.

$$h(x,y,z) = x^2y + z^3x + y^2z$$

$$\frac{\partial z}{\partial x} = \frac{-\partial h/\partial x}{\partial h/\partial z} = \frac{-(2xy + z^3)}{3z^2x + y^2}, \quad \left.\frac{\partial z}{\partial x}\right|_{(1,0,2)} = \frac{-8}{12} = \boxed{\frac{-2}{3}}$$

$$\frac{\partial z}{\partial y} = -\frac{\partial h/\partial y}{\partial h/\partial z} = \frac{-(x^2+2yz)}{3z^2x + y^2}, \quad \left.\frac{\partial z}{\partial y}\right|_{(1,0,2)} = \boxed{\frac{-1}{12}}$$

(b) Find the tangent plane equation to the surface at $(1, 0, 2)$.

$$z - 2 = \left.\frac{\partial z}{\partial x}\right|_{(1,0,2)}(x-1) + \left.\frac{\partial z}{\partial y}\right|_{(1,0,2)}(y-0)$$

$$\boxed{z - 2 = \frac{-2}{3}(x-1) - \frac{1}{12}y}$$

(c) Find an approximate solution for z to the equation $(0.9)^2(-0.1) + z^3(0.9) + (-0.1)^2 z = 8$ near $z = 2$ (Use the linear approximation).

$$(x_0, y_0, z_0) = (1, 0, 2)$$

$$dx = \Delta x = x - x_0 = 0.9 - 1 = -0.1$$

$$dy = \Delta y = y - y_0 = (-0.1) - 0 = -0.1$$

$$\Delta z \simeq dz = \left.\frac{\partial z}{\partial x}\right|_{(1,0,2)} dx + \left.\frac{\partial z}{\partial y}\right|_{(1,0,2)} dy$$

$$= \frac{-2}{3}(-0.1) - \frac{1}{12}(-0.1) = +0.075$$

$$z = z_0 + \Delta z \simeq z_0 + dz = 2 + 0.075$$

$$\boxed{= 2.075}$$

Q.5 (8+6 =14 pts) Suppose that $z = f(x, y)$ is a differentiable function on the xy-plane whose gradient field is given as $\nabla f(x, y) = (4x^3 - 4y)\,\mathbf{i} + (4y^3 - 4x)\,\mathbf{j}$.

(a) Find and classify all critical points of the function $f(x, y)$.

$$\left.\begin{array}{l} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{array}\right\} \quad \left.\begin{array}{l} y = x^3 \\ x = y^3 \end{array}\right\} \quad x = x^9$$

$$x^9 - x = 0$$

$$x(x^8 - 1) = 0$$

$$x(x^4 - 1)(x^4 + 1) = 0$$

$$x(x-1)(x+1)(x^2+1)(x^4+1) = 0$$

3 solns: $x = 0, 1, -1$

get $\boxed{(0,0), (-1,-1), (1,1)}$

$$f_{xx} = 12x^2, \quad f_{yy} = 12y^2, \quad f_{xy} = -4$$

$$D = f_{xx} f_{yy} - f_{xy}^2 = 144x^2y^2 - 16$$

At $(0,0)$: $D(0,0) = -16 < 0 \Rightarrow$ $\boxed{(0,0) \text{ is a saddle point.}}$

At $(-1,-1)$: $\left.\begin{array}{l} D(-1,-1) = 144 - 16 > 0 \\ f_{xx}(-1,-1) = 12 > 0 \end{array}\right\} \Rightarrow \boxed{(-1,-1) \text{ is a local min}}$

At $(1,1)$: $\left.\begin{array}{l} D(1,1) = 144 - 16 > 0 \\ f_{xx}(1,1) = 12 > 0 \end{array}\right\} \Rightarrow \boxed{(1,1) \text{ is a local min.}}$

(b) Find the directional derivative $D_{\mathbf{u}} f(1, 2)$, where $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

$$D_{\vec{u}} f(1,2) = \left(\nabla f \cdot \vec{u}\right)\Big|_{(1,2)}$$

$$= (-4, 28) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= \frac{24}{\sqrt{2}} = \boxed{12\sqrt{2}}$$

Q.6 (8×3 =24 pts) **(a)** Find $\int_1^2 \int_0^1 2xye^x dxdy$.

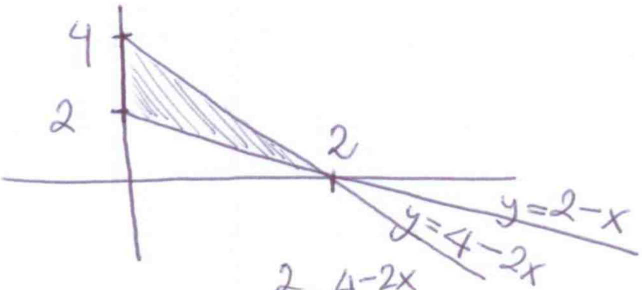
$$\int_1^2\int_0^1 2xye^x\,dx\,dy = \underbrace{\left(\int_1^2 dy\right)}_{1}\left(\int_0^1 2xe^x\,dx\right)$$

$u = x,\ dv = e^x dx$
$du = dx \quad v = e^x$

$$= xe^x\Big|_0^1 - \int_0^1 e^x\,du$$

$$= e - \left(e^x\Big|_0^1\right) = \boxed{1}$$

(b) Let D be the region bounded by the lines $y = 4 - 2x$, $y = 2 - x$ and $x = 0$ on the xy-plane. Find the volume of the solid lying over D and under the graph of $f(x, y) = 6xy$.



$$V = \iint_D 6xy\,dA = \int_0^2\int_{2-x}^{4-2x} 6xy\,dy\,dx$$

$$= \int_0^2\left(3xy^2\Big|_{2-x}^{4-2x}\right)dx$$

$$= \int_0^2 3x\left[(4-2x)^2-(2-x)^2\right]dx$$

$$= \int_0^2 3x\underbrace{(6-3x)(2-x)}_{12-12x+3x^2}\,dx$$

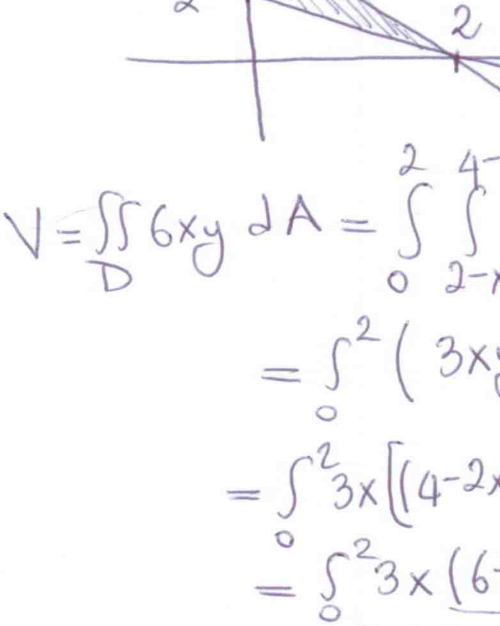
$$\to \int_0^2 36x-36x^2+9x^3\,dx$$

$$= 18x^2-12x^3+\frac{9}{4}x^4\Big|_0^2$$

$$= 72-96+36$$

$$= \boxed{12}$$

(c) Evaluate $\int_0^1 \int_{\sqrt{y}}^1 e^{x^3} dxdy$ by changing the integration order.



$$\int_0^1\int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy = \int_0^1\int_0^{x^2} e^{x^3}\,dy\,dx$$

$$= \int_0^1 x^2e^{x^3}\,dx$$

(let $u = x^3 \Rightarrow du = 3x^2dx$)

$$= \int_0^1 \frac{e^u}{3}\,du = \frac{e^u}{3}\Big|_0^1 = \boxed{\frac{e-1}{3}}$$