METU – NCC
Mathematics Group

	Calculus with Analytic Geometry Final Exam	
Code : MAT 120 Acad.Year : 2011-2012 Semester : Fall Instructors : Anar Dosiev Date : 13.01.2012 Time : 16.00 Duration : 100 minutes	Last Name : Name : KEY Dept. : Signature :	Stud. No : Sec. No :
	6 Questions on 4 Pages Total 100 Points	

1 (15)	2 (10)	3 (20)	4 (20)	5 (15)	6 (20)			

Q.1 (15 pts) Use the power series expansions to evaluate $\lim_{x\to 0} \frac{\sin(5x)}{1-e^{2x}}$. Note that

$$\frac{\sin(5x)}{1-e^{2x}} = \frac{5x - \frac{(5x)^3}{3!} + \frac{(5x)^5}{5!} - \cdots}{1 - 1 - 2x - \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} - \cdots} =$$

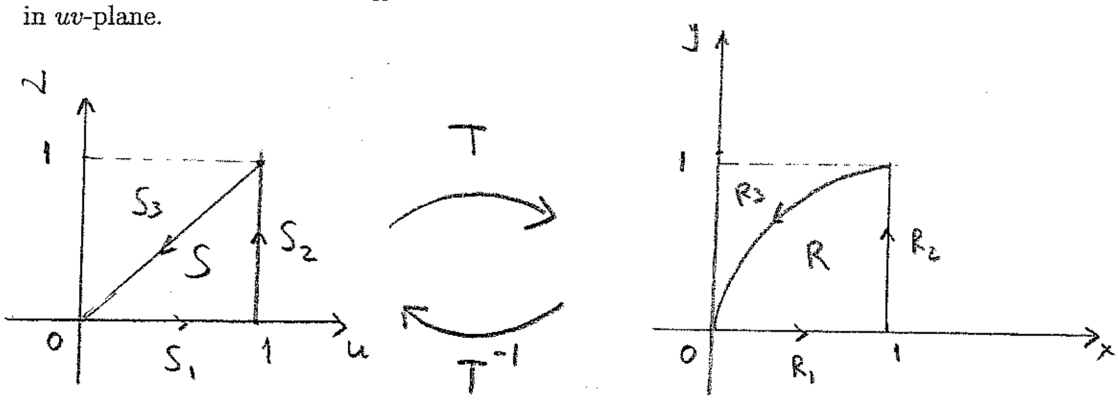
$$= -\frac{5}{2} \frac{1 - \frac{(5x)^2}{3!} + \frac{(5x)^4}{4!} - \cdots}{1 + \frac{2x}{2!} + \frac{(2x)^2}{3!} + \cdots}$$ whenever $x \neq 0$.

Whence $\lim_{x\to 0} \frac{\sin(5x)}{1-e^{2x}} = -\frac{5}{2}$

Q.2 (10 pts) State whether the equality $(\mathbf{a}\cdot\mathbf{b})\times\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is true or not for the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ in space. Explain your answer.

Not: $(\vec{a}\cdot\vec{b})\times\vec{c}$ is a "vector" in space whereas $\vec{a}\cdot(\vec{b}\times\vec{c})$ turns out to be a number (scalar)

Q.3 (20 pts) Sketch the region R in xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$ and $x = 1$. Use the transformation $T(u,v) = (x,y)$, $x = u^2$, $y = v$ to evaluate the double integral $\iint_R e^{y+\sqrt{x}}dA$. Sketch the relevant region $S = T^{-1}(R)$ in uv-plane.



$$T^{-1}(x,y) = (\underset{u}{\underset{\parallel}{\sqrt{x}}}, \underset{v}{\underset{\parallel}{y}})$$

Obviously, $S = T^{-1}(R)$ for the indicated S, and

$$J(T) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & 0 \\ 0 & 1 \end{vmatrix} = 2u > 0 \text{ inside of } S.$$

Therefore

$$\iint_R e^{y+\sqrt{x}}\,dA = \iint_S e^{v+u}\,2u\,du\,dv = 2\int_0^1 \left(\int_0^u u e^{u+v}dv\right)du$$

$$= 2\int_0^1 u e^u\left(\int_0^u e^v dv\right)du = 2\int_0^1 u e^u (e^u - 1)\,du =$$

$$= 2\int_0^1 u e^{2u}du - 2\int_0^1 u e^u du \overset{\text{Int. by Parts}}{=}$$

$$= u e^{2u}\Big|_0^1 - \int_0^1 e^{2u}du - 2\left(u e^u\Big|_0^1 - \int_0^1 e^u du\right) =$$

$$= e^2 - \frac{1}{2}(e^2 - 1) - 2e + 2(e-1) =$$

$$= \frac{e^2}{2} + \frac{1}{2} - 2 = \frac{e^2 - 3}{2}$$

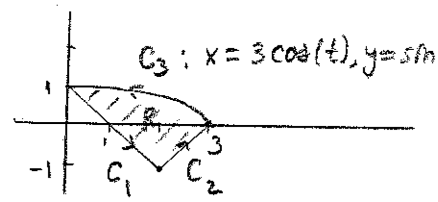
Q.4 (5+15=20 pts) Let R be a plane region with its area $A(R)$.

(a) Using Green's theorem prove the formula $A(R) = \frac{1}{2}\oint_C (-y dx + x dy)$, where C is the oriented boundary of the region R.

$$\frac{1}{2}\oint_C \underbrace{-y}_{P}\, dx + \underbrace{x}_{Q}\, dy = \frac{1}{2}\iint_R (Q_x - P_y)\, dA = \frac{1}{2}\iint_R 2\, dx\, dy = A(R)$$

(b) Sketch the plane region R enclosed by the ellipse through $(0,1)$ and $(3,0)$ centered at the origin; the line through the points $(0,1)$ and $(2,-1)$; the line through $(2,-1)$ and $(3,0)$. Find the the area $A(R)$ of this region based on (a).

Put $\omega = \frac{1}{2}(-y dx + x dy)$. Then



$$A(R) = \oint_C \omega = \int_{C_1}\omega + \int_{C_2}\omega + \int_{C_3}\omega =$$

$$= \frac{1}{2}\begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix} + \int_{C_3}\omega = -1 + \frac{3}{2} + \int_{C_3}\omega =$$

$$= \frac{1}{2} + \frac{1}{2}\int_0^{\pi/2}\left(3\sin^2(t) + 3\cos^2(t)\right)dt = \frac{1}{2} + \frac{1}{2}\int_0^{\pi/2} 3\, dt =$$

$$= \frac{1}{2} + \frac{3}{2}\,\frac{\pi}{2} = \frac{3\pi + 2}{4}.$$

Q.5 (15 pts) Find the power series expansion of the function $f(x) = x^2 e^{3x}$ about the point $a = 3$ (Don't use Taylor's formula). We have

$$f(x) = ((x-3)+3)^2 e^{3(x-3)+9} = e^9\left((x-3)^2 + 6(x-3) + 9\right)e^{3(x-3)}$$

$$= e^9\left[\sum_{n\geq 0}\frac{3^n}{n!}(x-3)^{n+2} + \sum_{n\geq 0}\frac{6\cdot 3^n}{n!}(x-3)^{n+1} + \sum_{n\geq 0}\frac{9\cdot 3^n}{n!}(x-3)^n\right]$$

$$= e^9\left[\sum_{n\geq 2}\frac{3^{n-2}}{(n-2)!}(x-3)^n + \sum_{n\geq 1}\frac{6\cdot 3^{n-1}}{(n-1)!}(x-3)^n + \sum_{n\geq 0}\frac{9\cdot 3^n}{n!}(x-3)^n\right]$$

$$= e^9\left[9 + \left(\frac{9\cdot 3}{1!} + 6\right)(x-3) + \sum_{n=2}^{\infty}\left(\frac{3^{n-2}}{(n-2)!} + \frac{6\cdot 3^{n-1}}{(n-1)!} + \frac{9\cdot 3^n}{n!}\right)(x-3)^n\right].$$

Q.6 (20 pts) Find the interval of convergence of the following power series $\sum_{n=3}^{\infty} \frac{(-1)^n \ln^9(n)}{n^2+2n-12}(2x+1)^n$. Write down all details.

The center is $a = -\frac{1}{2}$. If $u_n = \frac{(-1)^n \ln^9(n)}{n^2+2n-12}(2x+1)^n$

then we have $\rho_x = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| =$

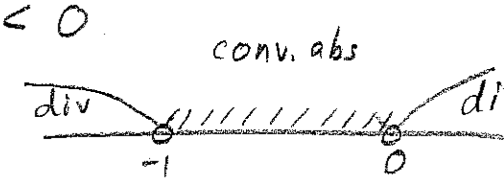
$$= \lim_{n\to\infty} \frac{\ln^9(n+1)\left((n+1)^2+2(n+1)-12\right)}{\ln^9(n)\,(n^2+2n-12)} |2x+1| =$$

$$= \lim_{n\to\infty} \left(\frac{\ln(n+1)}{\ln(n)}\right)^9 \frac{n^2+4n-9}{n^2+2n-12} |2x+1| = |2x+1|$$

if $\rho_x = |2x+1| < 1$ the series converges abs.

if $\rho_x > 1$ the series diverges.

But $|2x+1| < 1 \iff -1 < x < 0$



Now we have to look at the boundary points $x = -1$ and $x = 0$:

$$x = -1 \Rightarrow \sum_{n=3}^{\infty} \frac{\ln^9(n)}{n^2+2n-12} \overset{\text{lemma}}{\leq} \sum_{n=5}^{\infty} \frac{C n^{9\alpha}}{n^2-12} \leq \text{(for large } n)$$

$$\leq \sum_{n=5}^{\infty} \frac{C n^{9\alpha}}{n^2 - \frac{n^2}{2}} = \sum_{n=5}^{\infty} \frac{2C}{n^{2-9\alpha}}$$, pick $\alpha > 0$ so

that $2 - 9\alpha > 1$ or $0 < \alpha < \frac{1}{9}$ (for example, $\alpha = \frac{1}{10}$)

By p-Test, the series converges.

$$x = 0 \Rightarrow \sum_{n=3}^{\infty} \frac{(-1)^n \ln^9(n)}{n^2+2n-12}$$. But the series

converges abs.ly. By Abs. Conv. Test, the series converges conditionally too. Hence $I = [-1, 0]$.