Chapter 6

(4) Basis for

$$V = \{ p \in P_3 \mid p(0) = 0 \}$$

$$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + a_3 \cdot 0^3 = 0 \}$$

$$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0 = 0 \}$$

$$= \{ a_1 x + a_2 x^2 + a_3 x^3 \} = \text{Span} \{ x, x^2, x^3 \}$$

$$\boxed{\text{Basis} = \{ x, x^2, x^3 \}}$$

Basis for

$$S = \{ p \in P_3 \mid \tfrac{d}{dx} p(x) = 0 \}$$

$$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid \tfrac{d}{dx} (a_0 + a_1 x + a_2 x^2 + a_3 x^3) = 0 \}$$

$$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_1 + 2 a_2 x + 3 a_3 x^2 = 0 \}$$

$$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_1 = 0, \quad a_2 = 0, \quad a_3 = 0 \}$$

$$= \{ a_0 \} \qquad = \text{Span} \{ 1 \}$$

$$\boxed{\text{Basis} = \{ 1 \}}$$

Basis for

$$W = \{ p \in P_3 \mid p(-x) = p(x) \}$$

$$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0 + a_1 (-x) + a_2 (-x)^2 + a_3 (-x)^3 = a_0 + a_1 x + \cdots + a_3 x^3 \}$$

$$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0 = a_0, \ \underbrace{-a_1 = a_1}_{a_1 = 0}, \ a_2 = a_2, \ \underbrace{-a_3 = a_3}_{a_3 = 0} \}$$

$$= \{ a_0 + a_2 x^2 \} = \text{Span} \{ 1, x^2 \}$$

$$\boxed{\text{Basis} = \{ 1, x^2 \}}$$

(Recall: A function w/ $p(-x) = p(x)$ is "even"

⇝ for a polynomial, this means it has only even powers.

(8) **Prove:** If S is a subspace of f.d. vector space $\mathcal{V}$ then there is $T \subset \mathcal{V}$ with $S \cap T = \{\underline{0}\}$ and $S + T = \mathcal{V}$.

Proof:

Since $S \subset \mathcal{V}$ finite dimensional, S has a finite basis $\mathcal{B} = \{\underline{s}_1, \ldots, \underline{s}_k\}$. By the basis extension theorem, we can extend $\mathcal{B}$ to a basis $\{\underline{s}_1, \ldots, \underline{s}_k, \underline{t}_1, \ldots, \underline{t}_\ell\}$ of $\mathcal{V}$.

Let $T = \operatorname{Span}\{\underline{t}_1, \ldots, \underline{t}_\ell\}$. By construction,

$$\begin{aligned} S + T &= \operatorname{Span}\{\underline{s}_1, \ldots, \underline{s}_k\} + \operatorname{Span}\{\underline{t}_1, \ldots, \underline{t}_\ell\} \\ &= \operatorname{Span}\{\underline{s}_1, \ldots, \underline{s}_k, \underline{t}_1, \ldots, \underline{t}_\ell\} = \mathcal{V}. \end{aligned}$$

Suppose $\underline{v} \in S \cap T$. Then $\underline{v} \in \operatorname{Span}\{\underline{s}_1, \ldots, \underline{s}_k\}$ and $\underline{v} \in \operatorname{Span}\{\underline{t}_1, \ldots, \underline{t}_\ell\}$.

So $\underline{v} = a_1 \underline{s}_1 + \cdots + a_k \underline{s}_k$ and $\underline{v} = b_1 \underline{t}_1 + \cdots + b_\ell \underline{t}_\ell$. Thus

$$a_1 \underline{s}_1 + \cdots + a_k \underline{s}_k = \underline{v} = b_1 \underline{t}_1 + \cdots + b_\ell \underline{t}_\ell$$

$$0 = a_1 \underline{s}_1 + \cdots + a_k \underline{s}_k - b_1 \underline{t}_1 - \cdots - b_\ell \underline{t}_\ell$$

But $\{\underline{s}_1, \ldots, \underline{s}_k, \underline{b}_1, \ldots, \underline{b}_\ell\}$ is linearly indep. so $a_i = 0$ and $b_j = 0$.

Thus $\underline{v} = \underline{0}$, and so $S \cap T = \{\underline{0}\}$. ∎

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(14) Let $S = \{(x, y, z) \mid y - z = 0\}$. Find T so that $S \cap T = \{\underline{0}\}$, $S + T = \mathbb{R}^3$.

$$S = \{(x, y, y)\} = \operatorname{Span}\{(1,0,0), (0,1,1)\}$$

Basis for S: $\{(1,0,0), (0,1,1)\}$

To expand this to a basis for $\mathbb{R}^3$ we can add $(0,1,0)$.

(Then T will be $\operatorname{Span}\{(0,1,0)\} = \{(0, y, 0)\}$.)

$\operatorname{Span}\{(1,0,0), (0,1,1), (0,1,0)\} = \mathbb{R}^3$ because

$$(x, y, z) = x \cdot (1,0,0) + (z - y) \cdot (0,1,1) + y \cdot (0,1,0)$$

These are linearly indep. because

$$a \cdot (1,0,0) + b\,(0,1,1) + c \cdot (0,1,0) = 0 \rightsquigarrow \begin{cases} a = 0 \\ b + c = 0 \\ b = 0 \end{cases}$$

so $a = b = c = 0$.

(Note: A more standard choice

(10) **Prove:** If $S,T \subset V$ finite dimensional, then

$$\dim(S+T) = \dim(S) + \dim(T) - \dim(S\cap T)$$

Proof:

$S, T, S\cap T$ are all finite dimensional vector spaces.

Pick $\{\underline{v}_1,\dots,\underline{v}_k\}$ a basis for $S\cap T$. By the basis extension thm, we can extend $\{\underline{v}_1,\dots,\underline{v}_k\}$ to

$\{\underline{v}_1,\dots,\underline{v}_k,\underline{s}_1,\dots,\underline{s}_\ell\}$ a basis for S

and

$\{\underline{v}_1,\dots,\underline{v}_k,\underline{t}_1,\dots,\underline{t}_m\}$ a basis for T.

So $\dim(S) = k+\ell$
$\dim(T) = k+m$
$\dim(S\cap T) = k.$

Now we show $\dim(S+T) = k+\ell+m$
Idea: $S+T$ has basis $\{\underline{v}_1,\dots,\underline{v}_k,\underline{s}_1,\dots,\underline{s}_\ell,\underline{t}_1,\dots$

$$\begin{aligned} S+T &= \operatorname{Span}\{\underline{v}_1,\dots,\underline{v}_k,\underline{s}_1,\dots,\underline{s}_\ell\} + \operatorname{Span}\{\underline{v}_1,\dots,\underline{v}_k,\underline{t}_1,\dots,\underline{t}_m\} \\ &= \operatorname{Span}\{\underline{v}_1,\dots,\underline{v}_k,\underline{s}_1,\dots,\underline{s}_\ell,\underline{v}_1,\dots,\underline{v}_k,\underline{t}_1,\dots,\underline{t}_m\} \\ &= \operatorname{Span}\{\underline{v}_1,\dots,\underline{v}_k,\underline{s}_1,\dots,\underline{s}_\ell,\underline{t}_1,\dots,\underline{t}_m\}. \end{aligned}$$

Suppose $\sum_i a_i\underline{v}_i + \sum_i b_i\underline{s}_i + \sum c_i\underline{t}_i = 0$ then

$\sum_i a_i\underline{v}_i + \sum_i b_i\underline{s}_i = -\sum c_i\underline{t}_i$ is in S and T

Thus $\sum_i a_i\underline{v}_i + \sum_i b_i\underline{s}_i = -\sum c_i\underline{t}_i = \sum_i d_i\underline{v}_i$ for some d_i.

In particular $\sum_i (a_i - d_i)\underline{v}_i + \sum b_i\underline{s}_i = \underline{0}$

But $\{\underline{v}_1,\dots,\underline{v}_k,\underline{s}_1,\dots,\underline{s}_\ell\}$ is linearly indep. so $\begin{cases} a_i = d_i \\ b_i = 0 \end{cases}$

Similarly $\sum_i d_i\underline{v}_i + \sum_i c_i\underline{t}_i = 0$

so $\begin{cases} d_i = 0 \\ c_i = 0 \end{cases}$. Thus $\begin{cases} a_i = 0 \\ b_i = 0 \\ c_i = 0 \end{cases}$ so $\{\underline{v}_1,\dots,\underline{v}_k,\underline{s}_1,\dots,\underline{s}_\ell,\underline{t}_1,\dots,\underline{t}_m\}$ is linearly indep. ∎

⑰ <u>Prove</u>: If $\dim(\mathcal{V}) = n$ then $\{\underline{v}_1, \dots, \underline{v}_n\}$ is linearly indep. if and only if $\mathrm{Span}\{\underline{v}_1, \dots, \underline{v}_n\} = \mathcal{V}$.

<u>Proof:</u>

($\Rightarrow$) Suppose $\{\underline{v}_1, \dots, \underline{v}_n\}$ is linearly indep. If $\mathrm{Span}\{\underline{v}_1, \dots, \underline{v}_n\} \neq \mathcal{V}$ then we can extend to $\{\underline{v}_1, \dots, \underline{v}_n, \underline{w}_1, \dots, \underline{w}_k\}$ a basis for $\mathcal{V}$. Then $\dim(\mathcal{V}) = n+k > n$. Contradiction!

($\Leftarrow$) Suppose $\mathrm{Span}\{\underline{v}_1, \dots, \underline{v}_n\} = \mathcal{V}$. If $\{\underline{v}_1, \dots, \underline{v}_n\}$ not linearly indep. then it contains an independent subset

$$\{\underline{w}_1, \dots, \underline{w}_k\} \subset \{\underline{v}_1, \dots, \underline{v}_n\}$$

with the same span. $\mathrm{Span}\{\underline{w}_1, \dots, \underline{w}_k\} = \mathrm{Span}\{\underline{v}_1, \dots, \underline{v}_n\} = \mathcal{V}$. Then $\dim(\mathcal{V}) = k < n$. Contradiction! ◻

<u>Note:</u> If $\mathrm{Span}\{\underline{v}_1, \dots, \underline{v}_n\} = \mathcal{V}$ then $\dim(\mathcal{V}) \leq n$.
If $\{\underline{v}_1, \dots, \underline{v}_n\}$ is linearly indep. $\dim(\mathcal{V}) \geq n$.

㉓ Let $T \subset S$ a finite set. Find $\dim(\mathrm{Fun}(S,T))$.

Recall that $\mathrm{Fun}(S,T) = \{f: S \to \mathbb{R}$ with $f(t) = 0$ all $t \in T\}$.
We will use problem 9. (Remember that $S \setminus T = \{s \in S$ with $s \notin T\}$.)

$$\mathrm{Fun}(S,T) \cap \mathrm{Fun}(S, S \setminus T) = \mathrm{Fun}(S,S) = \{\underline{0}\}$$

(for f to be in the intersection above, it must vanish on T and $S \setminus T \Rightarrow$ it vanishes on all of S)

$$\mathrm{Fun}(S,T) + \mathrm{Fun}(S, S \setminus T) = \mathrm{Fun}(S)$$

Note that $\mathrm{Fun}(S, S \setminus T) \cong \mathrm{Fun}(T)$ (functions taking nonzero values on T)

$$\dim(\mathrm{Fun}(S,T)) = \dim(\mathrm{Fun}(S)) - \dim(\mathrm{Fun}(T))$$
$$= |S| - |T|.$$

(Maybe I should just have written the basis: $\{\chi_s\}_{s \in S \setminus T}$)

Chapter 7

(7) **Prove:** If $S \subset T$ finite dimensional with $\dim(S) = \dim(T)$ then $S = T$.

Proof:

Suppose $\dim(S) = \dim(T) = n$. Pick a basis $\{\underline{s}_1, \dots, \underline{s}_n\}$ of S.
Then $\{\underline{s}_1, \dots, \underline{s}_n\}$ is an independent set in T and $\dim(T) = n$.
By chapter 6 problem 17, $\text{Span}\{\underline{s}_1, \dots, \underline{s}_n\} = T$.
Therefore $S = T$. ∎

(9) Find a basis for $\{f \in P_4 \text{ with } f(z) = z^4 f(1/z)\} = V$.

$$V = \{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \text{ with } a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4 = z^4(a_0 + a_1(1/z) + a_2(1/z)^2 + a_3(1/z)^3 + \dots$$

$$= \{a_0 + \dots + a_4x^4 \text{ with } \boxed{a_0} + \boxed{a_1z} + a_2z^2 + a_3z^3 + a_4z^4 = a_0z^4 + a_1z^3 + a_2z^2 + \boxed{a_3z} + \boxed{a_4}\}$$

etc...

$$= \{a_0 + \dots + a_4x^4 \text{ with } \begin{array}{l} a_0 = a_4 \\ a_1 = a_3 \\ a_2 = a_2 \end{array}\}$$

$$= \{a_0 + a_1x + a_2x^2 + a_1x^3 + a_0x^4\}$$

$$= \text{Span}\{1 + x^4, x + x^3, x^2\}$$

$$\boxed{\text{Basis} = \{1 + x^4, x + x^3, x^2\}}$$

Chapter 8

⑥ Prove: If $T: \mathbb{R} \to \mathbb{R}$ then there is a number $a_T \in \mathbb{R}$ with

$$T(\underline{v}) = a_T \cdot \underline{v} \quad \text{for all } \underline{v} \in \mathbb{R}$$

Proof:

Let $a_T = T(\underline{1})$.

$$\begin{aligned} T(\underline{v}) &= T(v \cdot \underline{1}) \\ &= v\, T(\underline{1}) \\ &= v \cdot a_T \\ &= a_T \cdot \underline{v}. \end{aligned}$$

⑦ Suppose $T(3) = -4$. What is $T(-7)$?

$$\begin{aligned} T(-7) &= T(-\tfrac{7}{3} \cdot \underline{3}) \\ &= -\tfrac{7}{3} \cdot T(\underline{3}) \\ &= -\tfrac{7}{3} \cdot (-4) = \boxed{\tfrac{28}{3}.} \end{aligned}$$

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⑧ Prove: If $T: \mathbb{R}^2 \to \mathbb{R}$ then there are $a_T, b_T \in \mathbb{R}$ with

$$T(x,y) = a_T \cdot x + b_T \cdot y$$

Proof:

Let $a_T = T(1,0)$.
$b_T = T(0,1)$

$$\begin{aligned} T(x,y) &= T(x \cdot (1,0) + y \cdot (0,1)) \\ &= x\, T(1,0) + y \cdot T(0,1) \\ &= x \cdot a_T + y \cdot b_T \\ &= a_T x + b_T y \end{aligned}$$

⑨ Suppose $T(1,1) = 3$ and $T(1,0) = 4$. What is $T(2,1)$?

$$\begin{aligned} T(2,1) &= T((1,1) + (1,0)) \\ &= T(1,1) + T(1,0) \\ &= 3 + 4 = \boxed{7} \end{aligned}$$

(13) **Prove:** T injective if and only if $\ker(T) = \{\underline{0}\}$.

Proof:

($\Leftarrow$) Suppose $\ker(T) = \{\underline{0}\}$. If $T(\underline{v}) = T(\underline{w})$ then $\underline{0} = T(\underline{v}) - T(\underline{w}) = T(\underline{v} - \underline{w})$.

Since $\ker(T) = \{\underline{0}\}$, $\underline{v} - \underline{w} = \underline{0}$ so $\underline{v} = \underline{w}$. Thus T is injective.

($\Rightarrow$) Suppose T is injective. If $\underline{v} \in \ker(T)$ then $T(\underline{v}) = \underline{0} = T(\underline{0})$

Since T is injective, $\underline{v} = \underline{0}$. Thus $\ker(T) = \{\underline{0}\}$. ∎

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(17) **Prove:** the following are equivalent ($T : V \to V$)

(a) T is an isomorphism

(b) $\ker(T) = \{\underline{0}\}$

(c) $\operatorname{im}(T) = V$.

Proof:

By 8.6.1 (a) $\Leftrightarrow$ (b) and (c). It is enough to show (b) $\Leftrightarrow$ (c).

(b) $\Rightarrow$ (c): Suppose $\ker(T) = \{\underline{0}\}$. Let $\{\underline{e}_1, \dots, \underline{e}_n\}$ be a basis for V.

(We will show $\dim(\operatorname{Span}\{T\underline{e}_1, \dots, T\underline{e}_n\}) = n$. Since $\dim(V) = n$, this means $\operatorname{im}(T) = \operatorname{Span}\{T\underline{e}_1, \dots, T\underline{e}_n\} = V$.)

If $a_1 T\underline{e}_1 + \dots + a_n T\underline{e}_n = \underline{0}$ then

$$T(a_1\underline{e}_1 + \dots + a_n\underline{e}_n) = \underline{0} \quad \text{so} \quad a_1\underline{e}_1 + \dots + a_n\underline{e}_n \in \ker(T) = \{$$

But $\{\underline{e}_1, \dots, \underline{e}_n\}$ is linearly indep. so $a_1\underline{e}_1 + \dots + a_n\underline{e}_n = \underline{0}$ implies $a_1, \dots, a_n = 0$. Thus $\{T\underline{e}_1, \dots, T\underline{e}_n\}$ is linearly indep.

(c) $\Rightarrow$ (b): Suppose $\operatorname{im}(T) = V$. Let $\{\underline{e}_1, \dots, \underline{e}_n\}$ be a basis for V.

$\operatorname{Span}\{T\underline{e}_1, \dots, T\underline{e}_n\} = V$ and V has $\dim(V) = n$ so $\{T\underline{e}_1, \dots, T\underline{e}_n\}$ must be independent (see chapter 6 exercise 17)

If $\underline{v} \in \ker(T)$ then $T(\underline{v}) = \underline{0}$. Write $\underline{v} = a_1\underline{e}_1 + \dots + a_n\underline{e}_n$

$$\underline{0} = T(a_1\underline{e}_1 + \dots + a_n\underline{e}_n) = a_1 T(\underline{e}_1) + \dots + a_n T(\underline{e}_n)$$

But $\{T(\underline{e}_1), \dots, T(\underline{e}_n)\}$ are indep. so $a_1 = \dots = a_n = 0$.

Thus $\underline{v} = \underline{0}$.

(21) **Prove:** $\ker(T^2-1) = \ker(T+1) + \ker(T-1)$

Proof:

($\supset$) Note that $(T+1)\circ(T-1) = T\circ T - T\circ 1 + 1\circ T - 1$
$= T^2 - T + T - 1$
$= T^2 - 1.$

If $(T-1)\underline{v} = \underline{0}$ then $(T^2-1)\underline{v} = (T+1)\circ(T-1)\underline{v} = \underline{0}$

Thus $\ker(T-1) \subseteq \ker(T^2-1)$.

Similarly $(T-1)\circ(T+1) = T^2-1$ so $\ker(T+1) \subset \ker(T^2-1)$.

Therefore $\ker(T+1) + \ker(T-1) \subset \ker(T^2-1)$.

($\subset$) Note that $\frac{1}{2}(T+1)\underline{v} - \frac{1}{2}(T-1)\underline{v} = \underline{v}$

If $\underline{v} \in \ker(T^2-1)$ then

$$(T-1)\left(\tfrac{1}{2}(T+1)\underline{v}\right) = \tfrac{1}{2}(T^2-1)\underline{v} = 0$$
$$(T+1)\left(\tfrac{1}{2}(T-1)\underline{v}\right) = \tfrac{1}{2}(T^2-1)\underline{v} = 0$$

So $\underline{v} \in \ker(T-1) + \ker(T+1)$. ∎

(26) **Prove:** Let $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ be a basis for $\mathcal{V}$. Then <u>linear extension</u> gives an isomorphism of vector spaces

$$L : \mathrm{Fun}(\mathcal{B}, \mathcal{W}) \xrightarrow{\cong} \mathcal{L}(\mathcal{V}, \mathcal{W})$$

Proof:

I will skip the proof that linear extension is a linear transf.
→ it is easy: $\left(L(f+g)\right)(a_1\underline{b}_1 + \dots + a_n\underline{b}_n) = \left(L(f)\right)(a_1\underline{b}_1 + \dots + a_n\underline{b}_n) + \left(L(g)\right)(a_1\underline{b}_1 + \dots + a_n\underline{b}_n)$

$$\left(L(cf)\right)(a_1\underline{b}_1 + \dots + a_n\underline{b}_n) = c\left(L(f)\right)(a_1\underline{b}_1 + \dots + a_n\underline{b}_n).$$

L^{-1} is the "forgetting map"

$\mathcal{F} : \mathcal{L}(\mathcal{V}, \mathcal{W}) \longrightarrow \mathrm{Fun}(\mathcal{B}, \mathcal{W})$ defined by

$$\left(\mathcal{F}(f)\right)(\underline{b}_i) = f(\underline{b}_i)$$

(remember only what f did on the basis.)

Chapter 9

(8) Let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $T(x,y,z) = (y+z,\ x+z,\ x+y)$

Prove: T is an isomorphism.

Proof:

It is enough to show $\{T(1,0,0),\ T(0,1,0),\ T(0,0,1)\}$ is independent.

$$T(1,0,0) = (0,1,1)$$
$$T(0,1,0) = (1,0,1)$$
$$T(0,0,1) = (1,1,0)$$

$$a\begin{bmatrix}0\\1\\1\end{bmatrix} + b\begin{bmatrix}1\\0\\1\end{bmatrix} + c\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Longrightarrow \begin{cases} b+c=0 \\ a+c=0 \\ a+b=0 \end{cases} \quad b-a=0$$

$$\left.\begin{matrix} b-a=0 \\ a+b=0 \end{matrix}\right\rangle\ b=0 \quad \text{so} \quad a=0 \quad \text{and} \quad c=0. \quad \blacksquare$$

Find $T^{-1}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$.

T^{-1} is the linear extension of

$$\begin{cases} (0,1,1) \longmapsto T^{-1}(0,1,1) = (1,0,0) \\ (1,0,1) \longmapsto T^{-1}(1,0,1) = (0,1,0) \\ (1,1,0) \longmapsto T^{-1}(1,1,0) = (0,0,1) \end{cases}$$

$$T^{-1}(x,y,z) = T^{-1}\big(x\cdot(1,0,0) + y\cdot(0,1,0) + z(0,0,1)\big)$$

$$= T^{-1}\Big(x\cdot \tfrac{1}{2}\big((1,1,0)+(1,0,1)-(0,1,1)\big) + y\cdot\tfrac{1}{2}\big((1,1,0)+(0,1,1)-(1,0,1)\big) + z\cdot\tfrac{1}{2}\big((1,0,1)+(0,1,1)-(1,1,0)\big)\Big)$$

$$= \tfrac{1}{2}\Big(x\cdot\big((0,0,1)+(0,1,0)-(1,0,0)\big) + y\cdot\big((0,0,1)+(1,0,0)-(0,1,0)\big) + z\cdot\big((0,1,0)+(1,0,0)-(0,0,1)\big)\Big)$$

$$= \tfrac{1}{2}\big(-x+y+z,\ x-y+z,\ x+y-z\big)$$

(9) $\{\underline{e}_1, \dots, \underline{e}_n\}$ is a basis for $\mathcal{V}$. Define $\underline{e}_i^*: \mathcal{V} \to \mathbb{R}$ by

$$e_i^*(a_1\underline{e}_1 + \cdots + a_n\underline{e}_n) = a_i.$$

(1) Prove: $\underline{e}_i^*$ is a linear transformation

Proof:

$$\underline{e}_i^*\big((a_1\underline{e}_1 + \cdots + a_n\underline{e}_n) + (b_1\underline{e}_1 + \cdots + b_n\underline{e}_n)\big) = \underline{e}_i^*\big((a_1+b_1)\underline{e}_1 + \cdots + (a_n+b_n)\underline{e}_n\big)$$
$$= a_i + b_i$$
$$= e_i^*(a_1\underline{e}_1 + \cdots + a_n\underline{e}_n) + e_i^*(b_1\underline{e}_1 + \cdots + b_n\underline{e}_n)$$

$$e_i^*\big(c(a_1\underline{e}_1 + \cdots + a_n\underline{e}_n)\big) = e_i^*(ca_1\underline{e}_1 + \cdots + ca_n\underline{e}_n)$$
$$= c\cdot a_i$$
$$= c\cdot\underline{e}_i^*(a_1\underline{e}_1 + \cdots + a_n\underline{e}_n) \quad \blacksquare$$

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(2) Find a basis for $\ker(\underline{e}_i^*)$.

$$\ker(\underline{e}_1^*) = \{a_1\underline{e}_1 + \cdots + a_n\underline{e}_n \text{ with } \underline{e}_1^*(a_1e_1 + \cdots + a_n\underline{e}_n) = 0\}$$
$$= \{a_1\underline{e}_1 + \cdots + a_n\underline{e}_n \text{ with } a_1 = 0\}$$
$$= \{a_2\underline{e}_2 + \cdots + a_n\underline{e}_n\}.$$

→ In general, $\ker(\underline{e}_i^*) = \{a_1\underline{e}_1 + \cdots + a_{i-1}\underline{e}_{i-1} + a_{i+1}\underline{e}_{i+1} + \cdots + a_n e_n\}$

(no $a_i\underline{e}_i$)

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(3) Let $S: \mathcal{V} \to \mathcal{V}^*$ be the linear extension of $S(\underline{e}_i) = \underline{e}_i^*$.

Prove: S is an isomorphism.

Proof:

First show $\ker(S) = \{\underline{0}\}$: Suppose $\underline{0} = S(a_1\underline{e}_1 + \cdots + a_n\underline{e}_n)$
$$= a_1\underline{e}_1^* + \cdots + a_n\underline{e}_n^*$$

then $0 = \underline{0}(e_i) = (a_1 e_1^* + \cdots + a_n e_n^*)(e_i)$
$$= a_1 \cdot e_1^*(e_i) + \cdots + a_n \cdot e_n^*(e_i)$$
$$= a_i.$$

So $\ker(S) = \{\underline{0}\}$.

Now show $\operatorname{im}(S) = \mathcal{V}^*$: If $f \in \mathcal{V}^*$ then

$$f = f(\underline{e}_1)\cdot e_1^* + \cdots + f(\underline{e}_n)\cdot e_n^*$$

(4) Compute $\dim_{\mathbb{R}} \mathcal{L}(\mathcal{V}, \mathbb{R})$.

Since S is an isomorphism, $\{S(\underline{e}_1), \ldots, S(\underline{e}_n)\} = \{\underline{e}_1^*, \ldots, \underline{e}_n^*\}$ is a basis for $\mathcal{L}(\mathcal{V}, \mathbb{R}) = \mathcal{V}^*$.

$$\boxed{\dim_{\mathbb{R}} \mathcal{L}(\mathcal{V}, \mathbb{R}) = n}$$

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(15) Given $S, T \in \mathcal{V}^* = \mathcal{L}(\mathcal{V}, \mathbb{R})$ define $L : \mathcal{V} \longrightarrow \mathbb{R}^2$ by
$$L(\underline{v}) = (S\underline{v}, T\underline{v})$$

(1) **Prove:** L is a linear transf.

Proof:

$$\begin{aligned} L(\underline{v}+\underline{w}) &= (S(\underline{v}+\underline{w}), T(\underline{v}+\underline{w})) \\ &= (S\underline{v} + S\underline{w}, T\underline{v} + T\underline{w}) \\ &= (S\underline{v}, T\underline{v}) + (S\underline{w}, T\underline{w}) \\ &= L(\underline{v}) + L(\underline{w}). \end{aligned}$$

$$\begin{aligned} L(c\underline{v}) &= (S(c\underline{v}), T(c\underline{v})) \\ &= (cS\underline{v}, cT\underline{v}) \\ &= c(S\underline{v}, T\underline{v}) \\ &= c \cdot L\underline{v} \end{aligned}$$

∎

(2) **Prove:** $\ker(L) = \ker(S) \cap \ker(T)$.

Proof:

($\subset$) If $\underline{v} \in \ker(L)$ then $(S\underline{v}, T\underline{v}) = L\underline{v} = \underline{0} = (\underline{0}, \underline{0})$.

so $\underline{0} = S\underline{v}$ and $\underline{0} = T\underline{v}$.

($\supset$) If $\underline{v} \in \ker(S) \cap \ker(T)$ then $\begin{cases} 0 = S\underline{v} \\ 0 = T\underline{v} \end{cases}$

so $$\begin{aligned} \underline{0} &= (S\underline{v}, T\underline{v}) \\ &= L\underline{v}. \end{aligned}$$

∎

(16) **Prove:** If $C : \mathcal{V} \longrightarrow \mathbb{R}^2$ then $C\underline{v} = (C_1\underline{v}, C_2\underline{v})$ where $C_i : \mathcal{V} \to \mathbb{R}$.

Proof:

Homework.