



FIGURE 11

with center the origin and radius a , where a is chosen to be small enough that C' lies inside C . (See Figure 11.) Let D be the region bounded by C and C' . Then its positively oriented boundary is $C \cup (-C')$ and so the general version of Green's Theorem gives

$$\begin{aligned} \int_C P dx + Q dy + \int_{-C'} P dx + Q dy &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \iint_D \left[\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} \right] dA = 0 \end{aligned}$$

Therefore
$$\int_C P dx + Q dy = \int_{C'} P dx + Q dy$$

that is,
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$$

We now easily compute this last integral using the parametrization given by $\mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j}$, $0 \leq t \leq 2\pi$. Thus

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_0^{2\pi} \frac{(-a \sin t)(-a \sin t) + (a \cos t)(a \cos t)}{a^2 \cos^2 t + a^2 \sin^2 t} dt = \int_0^{2\pi} dt = 2\pi \end{aligned}$$

We end this section by using Green's Theorem to discuss a result that was stated in the preceding section.

SKETCH OF PROOF OF THEOREM 16.3.6 We're assuming that $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$ is a vector field on an open simply-connected region D , that P and Q have continuous first-order partial derivatives, and that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

If C is any simple closed path in D and R is the region that C encloses, then Green's Theorem gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R 0 dA = 0$$

A curve that is not simple crosses itself at one or more points and can be broken up into a number of simple curves. We have shown that the line integrals of \mathbf{F} around these simple curves are all 0 and, adding these integrals, we see that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C . Therefore $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D by Theorem 16.3.3. It follows that \mathbf{F} is a conservative vector field.

16.4 Exercises

1–4 Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

1. $\oint_C (x - y) dx + (x + y) dy$,
 C is the circle with center the origin and radius 2

2. $\oint_C xy dx + x^2 dy$,
 C is the rectangle with vertices $(0, 0)$, $(3, 0)$, $(3, 1)$, and $(0, 1)$
3. $\oint_C xy dx + x^2 y^3 dy$,
 C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$

4. $\oint_C x^2 y^2 dx + xy dy$, C consists of the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ and the line segments from $(1, 1)$ to $(0, 1)$ and from $(0, 1)$ to $(0, 0)$

5–10 Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

5. $\int_C xy^2 dx + 2x^2 y dy$,
 C is the triangle with vertices $(0, 0)$, $(2, 2)$, and $(2, 4)$
6. $\int_C \cos y dx + x^2 \sin y dy$,
 C is the rectangle with vertices $(0, 0)$, $(5, 0)$, $(5, 2)$, and $(0, 2)$
7. $\int_C (y + e^{\sqrt{x}}) dx + (2x + \cos y^2) dy$,
 C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$
8. $\int_C xe^{-2x} dx + (x^4 + 2x^2 y^2) dy$, C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$
9. $\int_C y^3 dx - x^3 dy$, C is the circle $x^2 + y^2 = 4$
10. $\int_C (1 - y^3) dx + (x^3 + e^{y^3}) dy$, C is the boundary of the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$

11–14 Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. (Check the orientation of the curve before applying the theorem.)

11. $\mathbf{F}(x, y) = \langle y \cos x - xy \sin x, xy + x \cos x \rangle$,
 C is the triangle from $(0, 0)$ to $(0, 4)$ to $(2, 0)$ to $(0, 0)$
12. $\mathbf{F}(x, y) = \langle e^{-x} + y^2, e^{-y} + x^2 \rangle$,
 C consists of the arc of the curve $y = \cos x$ from $(-\pi/2, 0)$ to $(\pi/2, 0)$ and the line segment from $(\pi/2, 0)$ to $(-\pi/2, 0)$
13. $\mathbf{F}(x, y) = \langle y - \cos y, x \sin y \rangle$,
 C is the circle $(x - 3)^2 + (y + 4)^2 = 4$ oriented clockwise
14. $\mathbf{F}(x, y) = \langle \sqrt{x^2 + 1}, \tan^{-1} x \rangle$, C is the triangle from $(0, 0)$ to $(1, 1)$ to $(0, 1)$ to $(0, 0)$

CAS 15–16 Verify Green's Theorem by using a computer algebra system to evaluate both the line integral and the double integral.

15. $P(x, y) = y^2 e^x$, $Q(x, y) = x^2 e^y$,
 C consists of the line segment from $(-1, 1)$ to $(1, 1)$ followed by the arc of the parabola $y = 2 - x^2$ from $(1, 1)$ to $(-1, 1)$
16. $P(x, y) = 2x - x^3 y^5$, $Q(x, y) = x^3 y^8$,
 C is the ellipse $4x^2 + y^2 = 4$

17. Use Green's Theorem to find the work done by the force $\mathbf{F}(x, y) = x(x + y)\mathbf{i} + xy^2\mathbf{j}$ in moving a particle from the origin along the x -axis to $(1, 0)$, then along the line segment to $(0, 1)$, and then back to the origin along the y -axis.
18. A particle starts at the point $(-2, 0)$, moves along the x -axis to $(2, 0)$, and then along the semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = \langle x, x^3 + 3xy^2 \rangle$.

19. Use one of the formulas in [5] to find the area under one arch of the cycloid $x = t - \sin t$, $y = 1 - \cos t$.

A 20. If a circle C with radius 1 rolls along the outside of the circle $x^2 + y^2 = 16$, a fixed point P on C traces out a curve called an *epicycloid*, with parametric equations $x = 5 \cos t - \cos 5t$, $y = 5 \sin t - \sin 5t$. Graph the epicycloid and use [5] to find the area it encloses.

21. (a) If C is the line segment connecting the point (x_1, y_1) to the point (x_2, y_2) , show that

$$\int_C x dy - y dx = x_1 y_2 - x_2 y_1$$

- (b) If the vertices of a polygon, in counterclockwise order, are (x_1, y_1) , (x_2, y_2) , \dots , (x_n, y_n) , show that the area of the polygon is

$$A = \frac{1}{2}[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \dots + (x_{n-1} y_n - x_n y_{n-1}) + (x_n y_1 - x_1 y_n)]$$

- (c) Find the area of the pentagon with vertices $(0, 0)$, $(2, 1)$, $(1, 3)$, $(0, 2)$, and $(-1, 1)$.

22. Let D be a region bounded by a simple closed path C in the xy -plane. Use Green's Theorem to prove that the coordinates of the centroid (\bar{x}, \bar{y}) of D are

$$\bar{x} = \frac{1}{2A} \oint_C x^2 dy \quad \bar{y} = -\frac{1}{2A} \oint_C y^2 dx$$

where A is the area of D .

23. Use Exercise 22 to find the centroid of a quarter-circular region of radius a .
24. Use Exercise 22 to find the centroid of the triangle with vertices $(0, 0)$, $(a, 0)$, and (a, b) , where $a > 0$ and $b > 0$.
25. A plane lamina with constant density $\rho(x, y) = \rho$ occupies a region in the xy -plane bounded by a simple closed path C . Show that its moments of inertia about the axes are

$$I_x = -\frac{\rho}{3} \oint_C y^3 dx \quad I_y = \frac{\rho}{3} \oint_C x^3 dy$$

26. Use Exercise 25 to find the moment of inertia of a circular disk of radius a with constant density ρ about a diameter. (Compare with Example 4 in Section 15.5.)
27. Use the method of Example 5 to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y) = \frac{2xy\mathbf{i} + (y^2 - x^2)\mathbf{j}}{(x^2 + y^2)^2}$$

and C is any positively oriented simple closed curve that encloses the origin.

28. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle x^2 + y, 3x - y^2 \rangle$ and C is the positively oriented boundary curve of a region D that has area 6.
29. If \mathbf{F} is the vector field of Example 5, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed path that does not pass through or enclose the origin.