M E T U – N C C
Mathematics Group

	Calculus II FINAL	
Code : MAT 120 Acad. Year: 2010-2011 Semester : Spring Instructor : S.A./E.G./H.T. Date : 30.05.2011 Time : 13.00 Duration : 120 minutes	Last Name : Name : Department: Signature :	Student No : Section No : **6** Questions on **6** Pages Total **100** Points

1	2	3	4	5	6		

Question 1 **(5 + 5 = 10 pts)**

(a) Use the geometric series $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $x \in (-1, 1)$ to evaluate the integral $\int \dfrac{dx}{1+x^5}$ as an infinite series.

$$\frac{1}{1+x^5} = \sum (-1)^n x^{5n}$$

$$\int \frac{dx}{1+x^5} = \sum \frac{(-1)^n x^{5n+1}}{5n+1}$$

(b) How many terms are needed to approximate the definite integral $\int_0^{1/2} \dfrac{dx}{1+x^5}$ within an error no more than $\dfrac{1}{100}$?

$$\int_0^{1/2} \frac{dx}{1+x^5} = \sum \frac{(-1)^n x^{5n+1}}{5n+1}\Bigg|_0^{1/2} = \sum \frac{(-1)^n}{2^{5n+1}(5n+1)}$$

For $n=1$: $\dfrac{1}{2^6 \cdot 6} < \dfrac{1}{100}$

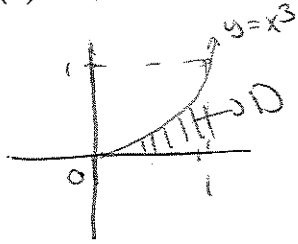
So 1 term is enough.

Question 2 ($4 \times 5 = 20$ **pts**)

(a) Evaluate the double integral $\int_2^3 \int_0^1 (ye^{xy} + 2x)\,dx dy$

$$\int_2^3 (e^{xy}+x^2)\Big|_0^1 dy = \int_2^3 e^y . dy = e^3 - e^2$$

(b) Evaluate the double integral $\displaystyle\int_0^1 \int_{y^{1/3}}^1 \frac{dxdy}{1+x^4}$



$$\iint_D \frac{dx\,dy}{1+x^4} = \int_0^1 \int_0^{x^3} \frac{1}{1+x^4} . dy . dx$$

$$= \int_0^1 \frac{x^3}{1+x^4} . dx = \frac{1}{4} \ln(1+x^4)\Big|_0^1$$

$$= \frac{1}{4} \ln 2$$

(c) Express the volume of a solid bounded by $x = 0$, $y = 0$, $2x+y+z = 4$ and $6x+3y-2z = 12$ as a double integral. Do **NOT** evaluate the integral.

Both planes have x & y -intercepts (2,0,0) & (0,4,0)

So the volum is

$$V = \iint_D (-2x - y - 4 - 3x - \tfrac{3}{2} y + 6)\, dA$$

where D is

$$V = \int_0^2 \int_0^{-2x+4} (-5x - \tfrac{5}{2} y + 2)\, dy dx$$

(d) Evaluate $\iint_{\mathcal{D}} 6\frac{y^3}{x^3}\,\mathrm{d}A$, where $\mathcal{D}$ is the region bounded by $xy = 1$, $xy = 4$, $y/x = 1$ and $y/x = 4$ in the first quadrant.

$u = xy$ & $v = \frac{y}{x}$ $\Rightarrow$ $x = \frac{\sqrt{u}}{\sqrt{v}}$ & $y = \sqrt{u}\sqrt{v}$

$T(u,v) = \left(\frac{\sqrt{u}}{\sqrt{v}}, \sqrt{u}\sqrt{v}\right)$ $\quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2\sqrt{u}\sqrt{v}} & -\frac{\sqrt{u}}{2\sqrt{v^3}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix}$

$xy = 1 \rightarrow u = 1$
$xy = 4 \rightarrow u = 4$
$y/x = 1 \rightarrow v = 1$
$y/x = 4 \rightarrow v = 4$

$= \frac{1}{2v}$

$$\iint_D 6\frac{y^3}{x^3}\,dA = \int_1^4\int_1^4 6v^3 \cdot \frac{1}{2v} \cdot du \cdot dv = \int_1^4 9v^2 \cdot dv = 3 \cdot (64) - 3$$
$$= 189$$

Question 3 (10 pts) Evaluate the line integral $\int_C f(x,y,z)\mathrm{d}s$ of the scalar field $f(x,y,z) = xy + y + z$ along the curve $\mathbf{r} = \quad 2t, t, 2-2t >$ for $t \in [0,1]$.

$$\int_C f(x,y,z)\,ds = \int_0^1 (2t^2 + t + 2 - 2t)\sqrt{2^2 + 1^2 + 2^2}\,.dt$$
$$= 3\int_0^1 (2t^2 - t + 2)dt = 3\left(\frac{2}{3}t^3 - \frac{1}{2}t^2 + 2t\right)\Big|_0^1$$
$$= 3\left(\frac{2}{3} - \frac{1}{2} + 2\right) = \frac{13}{2}$$

Question 4 **(10 + 10 + 5 = 25 pts)** Let $f(x,y) = 4x^2y + y^3 - 3y + 5$.

(a) Find and classify the critical points of $f(x,y)$.

$f_x = 8xy$

$f_y = 4x^2 + 3y^2 - 3$

$f_{xx} = 8y$

$f_{yy} = 6y$

$f_{xy} = 8x$

$D = 48y^2 - 64x^2$

$8xy = 0 \rightarrow x = 0$ or $y = 0$

$x=0 \downarrow y = \mp 1$, $\quad y=0 \downarrow x = \mp \frac{\sqrt{3}}{2}$

$D(0,1) > 0$, $f_{xx}(0,1) > 0$, $f(0,1) = 3$ Local min

$D(0,-1) > 0$, $f_{xx}(0,-1) < 0$, $f(0,-1) = 7$ Local max

$D(\frac{\sqrt{3}}{2}, 0) < 0$ Saddle pt

$D(-\frac{\sqrt{3}}{2}, 0) < 0$ Saddle pt

(b) Use the method of Langrange Multipliers to find the maximum and minimum values of $f(x,y)$ on the ellipse $x^2 + \frac{y^2}{4} = 1$.

$8xy = \lambda 2x$

$4x^2 + 3y^2 - 3 = \frac{\lambda y}{2}$

$x^2 + \frac{y^2}{4} = 1$

$\longrightarrow x = 0$ or $\lambda = 4y$

$x = 0 \downarrow y = \mp 2$

$\lambda = 4y \downarrow$

$4x^2 + 3y^2 - 3 = 2y^2$

$4x^2 + y^2 = 3$

$4x^2 + y^2 = 4$ $\Rightarrow$ ※.

$f(0,2) = 8 - 6 + 5 = 7 \leftarrow$ Max

$f(0,-2) = -8 + 6 + 5 = 3 \leftarrow$ Min

(c) Show that the double integral of $f(x,y)$ over the region $\mathcal{D} = \{(x,y) : 4x^2 + y^2 \leq 4\}$ satisfies the inequality $\iint_{\mathcal{D}} f(x,y)\, dA \leq 14\pi$.

Note. Recall that the area of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by πab.

By (a) & (b) $f(x,y) \leq 7$ on D. So

$\iint_D f(x,y)\, dA \leq 7 \cdot \text{Area}(D) = 14\pi$

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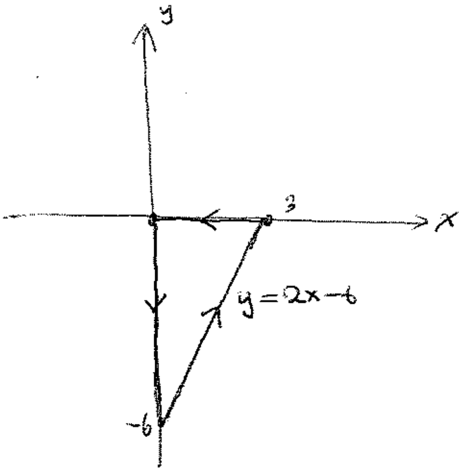
<u>Question 5</u> (15 pts) Use Green's theorem to evaluate the line integral

$$\oint_C [\sin(\tan x) - y^2]\mathrm{d}x + (5x + \mathrm{e}^{\cos y})\mathrm{d}y$$

where C is the positively oriented (counterclockwise) boundary of the triangular region with vertices $(0,0)$, $(0,-6)$ and $(3,0)$.



$$\oint \left(\sin(\tan x) - y^2\right) dx + \left(5x + e^{\cos y}\right) dy$$

$\Vert$ (Green's Theorem)

$$\iint_D (5+2y)\, dA = \int_0^3 \int_{2x-6}^{0} (5+2y)\, dy\, dx$$

$$= \int_0^3 \left(5y + y^2 \Big|_{(2x-6)}^{0} \right) dx = - \int_0^3 \left(5(2x-6) + (2x-6)^2 \right) dx$$

$$= - \int_0^3 (4x^2 - 14x + 6)\, dx = - \left(\frac{4}{3}x^3 - 7x^2 + 6x \Big|_0^3 \right)$$

$$= - [36 - 63 + 18] = 9$$

Question 6 (5 + 10 + 5 = 20 pts) Evaluate the line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ of the vector field $\mathbf{F}(x, y) = y\,\mathbf{i} + xy\,\mathbf{j}$

(a) Along the straightline extending from the point $A(0, -1)$ to $B(2, 1)$.

$r(t) = \langle t, t-1 \rangle \quad , \quad 0 \leq t \leq 2$

$\int_C F \cdot dr = \int_0^2 \langle t-1, t^2-t \rangle \cdot \langle 1, 1 \rangle \, dt = \int_0^2 (t^2-1)dt = \frac{t^3}{3} - t \Big|_0^2$

$= \frac{8}{3} - 2$

$\frac{x^2}{2} - 1$

(b) Along the cubic $y = \frac{1}{4}x^3 - 1$ extending from the point $A(0, -1)$ to $B(2, 1)$.

$r(t) = \langle t, \frac{1}{4}t^3 - 1 \rangle \quad , \quad 0 \leq t \leq 2$

$\int_C F \cdot dr = \int_0^2 \langle \frac{1}{4}t^3 - 1, \frac{1}{4}t^4 - t \rangle \langle 1, \frac{3}{4}t^2 \rangle \, dt$

$= \int_0^2 \left(\frac{1}{4}t^3 - 1 + \frac{3}{16}t^6 - \frac{3}{4}t^3 \right) dt$

$= -\frac{1}{8}t^4 + \frac{3}{16 \cdot 7}t^7 - t \Big|_0^2 = -2 + \frac{24}{7} - 2$

(c) By inspecting the results of Part(**a**) and Part(**b**), can you determine whether **F** is **conservative** or not?

Answers in (a) & (b) are different so $\int F \cdot dr$ is not path independent. Then F is <u>not</u> conservative