M E T U
Northern Cyprus Campus

	Calculus for Functions of Several Variables I. Midterm	
Code : *Math 120* Acad.Year: *2010-2011* Semester : *Spring* Date : *3.26.2011* Time : *10:00* Duration : *120 minutes*	Last Name: Name : Department: Signature:	Student No: Section:
	6 QUESTIONS ON 6 PAGES TOTAL 100 POINTS	

1	2	3	4	5	6		

1. (5 pts each) Find $\lim_{n\to\infty} a_n$ where the general term a_n is as below:

(a) $a_n = \frac{\cos(n)-1}{n}$

$$\frac{-2}{n} \leqslant \frac{\cos(n)-1}{n} \leqslant \frac{0}{n} \quad \Rightarrow \lim_{n\to\infty} \frac{-2}{n} = \lim_{n\to\infty} 0 = 0$$

Using squeeze Theorem; $\lim_{n\to\infty} \frac{\cos(n)-1}{n} = 0$

(b) $a_n = n(e^{1/n}-1)$

$$\lim_{n\to\infty} n(e^{1/n}-1) = \lim_{n\to\infty} \frac{e^{1/n}-1}{1/n} = \lim_{x\to\infty} \frac{e^{1/x}-1}{1/x} \overset{L'H.R}{=} \lim_{x\to\infty} \frac{e^{1/x}\cdot \cancel{-1/x^2}}{\cancel{-1/x^2}} =$$

(c) $a_n = \frac{(-1)^n}{\cos(\pi n)}$

$$\lim_{n\to\infty} \frac{(-1)^n}{\cos(\pi n)} = \lim_{n\to\infty} \frac{\cancel{(-1)^n}}{\cancel{(-1)^n}} = 1.$$

(d) $a_n = \frac{1}{\sin(n)}$

$$\lim_{n\to\infty} \frac{1}{\sin(n)} = \frac{1}{\underbrace{\lim_{n\to\infty} \sin(n)}_{D.N.E}}$$

So, limit does not exist.

2. (5 pts eachs) Determine whether the following series converge or diverge (Specify clearly the method/test used and give details).

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{3^n (n!)^2}$

Using Ratio Test: $\displaystyle\lim_{n\to\infty} \left| \frac{\frac{(2n+2)!}{3^{n+1}((n+1)!)^2}}{\frac{(2n)!}{3^n (n!)^2}} \right| = \lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)}{3(n+1)^2} \right| =$

So, the series is divergent

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n)}{\sqrt{n^3+1}}$

Let's check absolute convergence; $\left|\frac{\cos(n)}{\sqrt{n^3+1}}\right| < \frac{1}{n^{3/2}}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is p-series $(p = 3/2 > 1)$, then it is convergent.

By Comparison Test, our series is absolutely convergent, and henc

our series is convergent.

(c) $\displaystyle\sum_{n=1}^{\infty} n\sin(1/n)$

Using Test for Divergence; $\displaystyle\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$

So, it is divergent

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(3n)!}$

Using Alternating Series Test; the series is convergent.

- $\frac{1}{(3(n+1))!} < \frac{1}{(3n)!}$ ✓
- $\displaystyle\lim_{n\to\infty} \frac{1}{(3n)!} = 0$ ✓

(e) $\sum_{n=0}^{\infty}\left(\frac{4+3n}{1+2n}\right)^n$

Using Root Test: $\lim_{n\to\infty} \sqrt[n]{\left(\frac{4+3n}{1+2n}\right)^n} = \lim_{n\to\infty} \frac{4+3n}{1+2n} = \frac{3}{2} > 1$.

So, the series is divergent.

3. (10+5 pts) Let $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-3)^n}{n^2 3^n}$

(a) Find the domain of $f(x)$ (i.e., the interval of convergence of the series).

Using Ratio Test $\lim_{n\to\infty} \left| \frac{\frac{(-1)^n (x-3)^{n+1}}{(n+1)^2 3^{n+1}}}{\frac{(-1)^{n-1}(x-3)^n}{n^2 3^n}} \right| = \lim_{n\to\infty} \left| \frac{n^2 (x-3)}{(n+1)^2 \cdot 3} \right| = \frac{|x-3|}{3}$

So, our series is convergent when $\frac{|x-3|}{3} < 1 \Rightarrow 0 < x < 6$

We still need to check end points.

At $x=0$; we have $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-3)^n}{n^2 3^n} = -\sum_{n=1}^{\infty} \frac{1}{n^2}$. It is p-series $(p=2>1$
so, it is convergent

At $x=6$; we have $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cancel{3^n}}{n^2 \cancel{3^n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$. It is Alternating series.

- $\frac{1}{(n+1)^2} < \frac{1}{n^2}$ ✓ It is als
- $\lim_{n\to\infty} \frac{1}{n^2} = 0$ ✓ converg

Thus, domain of $f(x)$ is: $[0,6]$.

(b) Find $f^{(2011)}(3)$.

Let's use Taylor Series expansion of $f(x)$ around 3.
We know that n'th coefficient of the series is:

$$C_n = \frac{f^{(n)}(3)}{n!} \quad \text{so,} \quad C_{2011} = \frac{f^{(2011)}(3)}{2011!}$$

where in the series C_n is given by $\frac{(-1)^{n-1}}{n^2 3^n}$, then $C_{2011} = \frac{(-1)^{20}}{(2011)^2 \cdot 3^2}$

$$\Rightarrow \frac{f^{(2011)}(3)}{(2011)!} = \frac{1}{(2011)^2 \cdot 3^{2011}} \Rightarrow f^{(2011)}(3) = \frac{(2010)!}{2011 \cdot 3^{2011}}$$

4. (5 pts each) Find the first 3 nonzero terms of the Taylor Series expansion of the functions around the indicated points below.

(a) $e^x \cos(x)$ around 0.

Remember $\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$

So, $e^x . \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$

$$e^x \cos x = 1 + x - \left(\frac{x^3}{2!} - \frac{x^3}{3!}\right) + \left(\frac{x^4}{4!} - \frac{x^4}{(2!)^2} + \frac{x^4}{4!}\right) + \cdots$$

$$e^x \cos x = \underbrace{1 + x - \frac{x^3}{3}}_{\text{first nonzero 3 terms}} + \cdots$$

(b) $\frac{1}{\sqrt{3x-2}}$ around 2.

$$\frac{1}{\sqrt{3x-2}} = \frac{1}{\sqrt{3(x-2)+4}} = \frac{1}{2} \cdot \left(1 + \frac{3(x-2)}{4}\right)^{-\frac{1}{2}}$$

Remember Binomial expansion;

$$\frac{1}{\sqrt{3x-2}} = \frac{1}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left[\frac{3(x-2)}{4}\right]^n \qquad \left|\frac{3(x-2)}{4}\right| <$$

$$= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{3^n (x-2)^n}{2^{2n+1}}, \qquad |x-2| < \frac{4}{3}$$

$$= \underbrace{\frac{1}{2} - \frac{3(x-2)}{16} + \frac{27(x-2)^2}{256}}_{\text{first nonzero 3 terms}} \cdots$$

5. (10+5 pts)

(a) Find the power series expansion of $\frac{x^3}{(1-x)^3}$. [Recall that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.]

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow \frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \sum_{n=0}^{\infty} n.x^{n-1}$$

$$\Rightarrow \frac{2}{(1-x)^3} = \left(\frac{1}{1-x}\right)'' = \sum_{n=0}^{\infty} n(n-1)\, x^{n-2}$$

So, if we multiply both siden by $\frac{x^3}{2}$ we get;

$$\frac{x^3}{(1-x)^3} = \sum_{n=0}^{\infty} \frac{n(n-1)}{2}\, x^{n+1}$$

(b) Use part (a) to find the exact value of $\sum_{n=2}^{\infty} \frac{n^2-n}{2^{n+1}}$.

Using part (a); $\sum_{n=0}^{\infty} \frac{n^2-n}{2^{n+1}} = 2.\sum_{n=0}^{\infty} \frac{n^2-n}{2}.x^{n+1} \Big|_{x=\frac{1}{2}}$

$$= 2\cdot\left(\frac{x^3}{(1-x)^3}\right)\Big|_{x=\frac{1}{2}}$$

$$= 2.$$

6. (10+5 pts) Consider the definite integral $\int_0^1 \frac{dx}{16+x^4}$.

(a) Find the value of this definite integral as an infinite series.

$$\frac{1}{16+x^4} = \frac{1}{16} \cdot \frac{1}{1-\left(-\left(\frac{x}{2}\right)^4\right)} = \frac{1}{16} \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{4n}}{2^{4n}}, \quad \left|\frac{x}{2}\right| < 1$$

$\left(\text{remember } \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\right)$

Our integral becomes;

$$\int_0^1 \frac{1}{16} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{2^{4n}} dx = \frac{1}{16} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{2^{4n}(4n+1)} \Bigg|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n+4} \cdot (4n+1)}$$

(b) How many terms of the series are needed to approximate the value of the integral within an error less than 0.0001.

For Alternating Series, taking first N terms having an error less than the next term.

$$|\text{Error}| < \frac{1}{2^{4(N+1)+4} \cdot (4(N+1)+1)} < \frac{1}{10000}$$

if $N=1$, $\frac{1}{2^{12} \cdot 9} < \frac{1}{10.000}$ is satis

So, $\int_0^1 \frac{dx}{16+x^4} \approx \frac{1}{2^4} - \frac{1}{2^8 \cdot 5}$ with an error less than 0.0001.