M E T U
Northern Cyprus Campus

	Calculus for Functions of Several Variables I. Midterm	
Code : *Math 120* Acad.Year: *2009-2010* Semester : *Spring* Date : *27.3.2010* Time : *9:00* Duration : *120 minutes*	Last Name: KEY Name : Department: Signature:	Student No Section:
	6 QUESTIONS ON 7 PAGES TOTAL 100 POINTS	

1	2	3	4	5	6		

1. (5+5+5=15 pts.) Determine whether the following sequences converge or diverge. Find the limits of the ones which converge.

(a) $a_n = \dfrac{\sin n}{n}$

$$-1 \leqslant \sin n \leqslant 1 \Rightarrow \frac{-1}{n} \leqslant \frac{\sin n}{n} \leqslant \frac{1}{n}$$

$\lim_{n\to\infty}\left(\frac{-1}{n}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$. So, by squeeze theorem, $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

(b) $a_n = \ln\left(\dfrac{n}{n^2+1}\right)$

$\lim_{n\to\infty} \frac{n}{n^2+1} = 0$. Since $\ln 0$ is undefined, $\lim_{n\to\infty} a_n$ does not exist.

(c) $a_n = \left(\dfrac{n}{n+1}\right)^n$

$$\ln a_n = n \ln\left(\frac{n}{n+1}\right) = n(\ln n - \ln(n+1))$$

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \underbrace{n \cdot \ln\left(\frac{n}{n+1}\right)}_{\boxed{\infty \cdot 0}} = \lim_{n\to\infty} \frac{(\ln n - \ln(n+1))}{1/n} \quad \boxed{\frac{\infty}{\infty}}$$

$$\overset{\text{L'Hospital}}{=} \lim_{n\to\infty} \frac{\frac{1}{n} - \frac{1}{n+1}}{-1/n^2}$$

$$= \lim_{n\to\infty} -\frac{n^2}{n^2+n} = -1$$

By continuity of $\ln$ at $\frac{1}{e}$, $\lim_{n\to\infty} \ln a_n = \ln \lim_{n\to\infty} a_n = -1$

$$\Rightarrow \lim a_n = e^{-1} = 1/e$$

2. (8 pts.) Say $a_{n+1} = \dfrac{a_n + 2}{2}$ for $n \geq 1$ and $a_1 = 10$.

(a) Show that $\{a_n\}$ is bounded below.

Let us show that $a_n \geqslant 2$ for all n.
$a_1 = 10 > 2$.
Suppose that $a_n \geqslant 2$. Then

$$a_{n+1} = \frac{a_n + 2}{2} \geqslant \frac{2+2}{2} = 2.$$

Hence, by induction, $a_n \geqslant 2$ for all n.

(b) Show that $\{a_n\}$ is decreasing.

$$a_2 = \frac{10+2}{2} = 6 \leqslant a_1 = 10$$

Suppose that $a_n \leqslant a_{n-1}$ Then,

$$a_n + 2 \leqslant a_{n-1} + 2$$

$$\Rightarrow a_{n+1} = \frac{a_n + 2}{2} \leqslant a_n = \frac{a_{n-1} + 2}{2}$$

By induction, $\{a_n\}$ is decreasing.

(c) Does $\lim_{n\to\infty} a_n$ exist? If so, find its value.

Since $\{a_n\}$ is monotone decreasing and bounded below, by the monotone convergence theorem, $\lim\limits_{n\to\infty} a_n$ exists. Suppose $\lim\limits_{n\to\infty} a_n = L$

$$a_n = \frac{a_{n-1} + 2}{2}$$

$$\lim_{n\to\infty} a_n = \frac{\lim\limits_{n\to\infty} a_{n-1} + 2}{2}$$

$$L = \frac{L+2}{2} \Rightarrow L = 2.$$

3. (5+5+5+5+5=25 pts.) Determine whether the following series converge or diverge (Specify clearly the method/test used, and give details).

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n^2+2n}}{n^4-3n^2-1}$.

Apply the limit comparison test with $\frac{1}{n^3}$

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n^2+2n}}{n^4-3n^2-1}}{1/n^3} = \lim_{n\to\infty} \frac{n^4\sqrt{1+2/n}}{n^4-3n^2-1} = 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by the p-test, $\sum_{n=2}^{\infty} \frac{\sqrt{n^2+2n}}{n^4-3n^2-1}$ converges

(note that $n^4-3n^2-1 = n^2(n^2-3)-1 > 0$ for $n \geq 2$)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\arctan n}$.

$$\lim_{n\to\infty} \frac{1}{\arctan n} = \frac{1}{\pi/2} = \frac{2}{\pi} \neq 0.$$

So, by the general term test, $\sum_{n=1}^{\infty} \frac{1}{\arctan n}$ diverges.

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\ln n}{\ln(n^2+1)}\right)^n$.

Apply the root test:

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{\ln n}{\ln(n^2+1)}\right)^n} = \lim_{n\to\infty} \frac{\ln n}{\ln(n^2+1)} \quad \boxed{\frac{\infty}{\infty}}$$

$$\stackrel{\text{L'Hospital}}{=} \lim_{n\to\infty} \frac{1/n}{2n/(n^2+1)}$$

$$= \lim_{n\to\infty} \frac{n^2+1}{2n^2}$$

$$= \frac{1}{2} < 1$$

Hence, by the root test, $\sum_{n=1}^{\infty} \left(\frac{\ln n}{\ln(n^2+1)}\right)^n$ converges.

(d) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\sqrt{\ln n}}$.

Apply the integral test. Both x and $\sqrt{\ln x}$ are monotone increasing, so $\frac{1}{x\sqrt{\ln x}}$ is decreasing and $\frac{1}{n\sqrt{\ln n}} \geq 0$ for $n \geq 3$, so we can apply the integral test

$$\int_3^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{M\to\infty} \int_3^{M} \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{M\to\infty} \int_{\ln 3}^{\ln M} \frac{du}{\sqrt{u}}$$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx \end{aligned}}$$

$$= \lim_{M\to\infty} \left.\frac{u^{1/2}}{1/2}\right|_{\ln 3}^{\ln M}$$

$$= \lim_{M\to\infty} \left(2\sqrt{\ln M} - 2\sqrt{\ln 3}\right)$$

$$= \infty \quad (\text{diverges})$$

(e) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{1+\frac{1}{n}} - \sqrt{1-\frac{1}{n}}\right)$.

$$\sqrt{1+\frac{1}{n}} - \sqrt{1-\frac{1}{n}} = \frac{\left(\sqrt{1+\frac{1}{n}} - \sqrt{1-\frac{1}{n}}\right)\left(\sqrt{1+\frac{1}{n}} + \sqrt{1-\frac{1}{n}}\right)}{\left(\sqrt{1+\frac{1}{n}} + \sqrt{1-\frac{1}{n}}\right)}$$

$$= \frac{2/n}{\sqrt{1+\frac{1}{n}} + \sqrt{1-\frac{1}{n}}}$$

Apply limit comparison test with $1/n$

$$\lim_{n\to\infty} \frac{\frac{2/n}{\sqrt{1+\frac{1}{n}} + \sqrt{1-\frac{1}{n}}}}{1/n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}} + \sqrt{1-\frac{1}{n}}}$$

$$= \frac{1}{2}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series), $\displaystyle\sum_{n=1}^{\infty} \sqrt{1+\frac{1}{n}} - \sqrt{1-\frac{1}{n}}$ diverges.

(Note that $1+\frac{1}{n} > 1-\frac{1}{n} \Rightarrow \sqrt{1+\frac{1}{n}} - \sqrt{1-\frac{1}{n}} > 0$)

4. (12 pts.) Find the radius and the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2^n(2x-3)^n}{\sqrt{n}3^n}$$

Apply the ratio test:

$$\lim_{n\to\infty} \left| \frac{\dfrac{2^{n+1}(2x-3)^{n+1}}{\sqrt{n+1}\;3^{n+1}}}{\dfrac{2^n(2x-3)^n}{\sqrt{n}\;3^n}} \right| = \lim_{n\to\infty} \left| \frac{2}{3}(2x-3)\frac{\sqrt{n}}{\sqrt{n+1}} \right|$$

$$= \frac{2}{3}|2x-3|$$

So the series converges if $\frac{2}{3}|2x-3| < 1$ and diverges if $\frac{2}{3}|2x-3| > 1$.

$$\frac{2}{3}|2x-3| < 1 \iff |2x-3| < \frac{3}{2}$$

$$\iff -\frac{3}{2} < 2x-3 < \frac{3}{2}$$

$$\iff \frac{3}{2} < 2x < \frac{9}{2}$$

$$\iff \frac{3}{4} < x < \frac{9}{4}$$

The radius of convergence is $\frac{1}{2}\left(\frac{9}{4} - \frac{3}{4}\right) = \frac{3}{4}$

$\underline{x = \frac{3}{4}}$: $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}\,3^n}\cdot\left(\frac{-3}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ alternating series with $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and $\frac{1}{\sqrt{n}}$ decreasing

so, by the alternating series test, series converges

$\underline{x = \frac{9}{4}}$: $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}\,3^n}\left(\frac{3}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the p-test ($p = 1/2$)

Interval of convergence: $[3/4, 9/4)$

5. (5+5+5+5+5=25 pts.) Find the following power series expansions:

(a) $\dfrac{2x}{3+4x^2}$ around $a=0$.

$$\frac{1}{3+4x^2} = \frac{1}{3}\cdot\frac{1}{1-\left(-\frac{4x^2}{3}\right)} = \frac{1}{3}\left(1-\frac{4x^2}{3}+\frac{16}{9}x^4 - \cdots\right)$$

$$\Rightarrow \frac{2x}{3+4x^2} = \sum_{n=0}^{\infty} \frac{2}{3}\cdot(-1)^n\left(\frac{4}{3}\right)^n x^{2n+1}$$

(b) $3xe^{-x^2}$ around $a=0$.

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \Rightarrow e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$$

$$3xe^{-x^2} = \sum_{n=0}^{\infty}\frac{(-1)^n 3x^{2n+1}}{n!}$$

(c) $\arctan(x^2)$ around $a=0$.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\arctan x = \int_0^x \frac{1}{1+x^2}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} \Rightarrow \arctan(x^2) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{2n+1}$$

(d) $\sin x$ around $a=\pi$.

$f(x)=\sin x$ $\qquad f(\pi)=0$

$\Rightarrow f'(x)=\cos x$ $\qquad f'(\pi)=-1$

$f''(x)=-\sin x$ $\qquad f''(\pi)=0$

$f'''(x)=-\cos x$ $\qquad f'''(\pi)=1$

$f^{(4)}(x)=\sin x$ $\qquad f^{(4)}(\pi)=0$

$\vdots \qquad \vdots$

$$\sin x = \sum_{n=0}^{\infty}\frac{f^{(n)}(\pi)}{n!}(x-\pi)^n$$

$$= -(x-\pi)+\frac{(x-\pi)^3}{3!}-\frac{(x-\pi)^5}{5!} + \cdots$$

(e) $\dfrac{1}{1-2x}$ around $a=2$.

$$\frac{1}{1-2x} = \frac{1}{1-2(x-2)-4} = \frac{1}{-3-2(x-2)}$$

$$= \frac{-1}{3}\cdot\frac{1}{1+\frac{2}{3}(x-2)}$$

$$= -\frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{2}{3}(x-2)\right)^n$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{2^n}{3^{n+1}}(x-2)^n$$

6. (15 pts.) (a) Find the power series expansion of $\dfrac{1}{\sqrt{1-x^2}}$ around $a = 0$, and write the first three nonzero terms explicitly.

By the binomial theorem,

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n \cdot x^{2n}$$

$$= 1 + \frac{x^2}{2} + \frac{3}{8} x^4 + \cdots$$

(b) Use the relation $\int_0^t \frac{1}{\sqrt{1-x^2}} dx = \arcsin t$ to find the power series expansion of $\arcsin x$ around $a = 0$.

$$\arcsin t = \int_0^t \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n x^{2n} \, dx$$

$$= \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n \frac{t^{2n+1}}{2n+1}$$

$$\Rightarrow \arcsin x = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n \frac{x^{2n+1}}{2n+1} = x + \frac{x^3}{6} + \frac{3}{40} x^5 + \cdots$$

(c) Use the first three nonzero terms of $6 \arcsin x$ at $x = \frac{1}{2}$ to estimate π.

$$6 \arcsin x = 6x + x^3 + \frac{18}{20} x^5 + \cdots$$

$$\pi = \underbrace{6 \arcsin \frac{1}{2}}_{\pi/6} = 6 \cdot \frac{1}{2} + \frac{1}{8} + \frac{\cancel{18}^{9}}{40} \cdot \frac{1}{\cancel{32}_{16}} + \cdots$$

$$= 3 + 0.125 + 0.0140 + \cdots$$

$$\simeq 3.139$$