it seems plausible that the *exact* distance d traveled is the *limit* of such expressions:

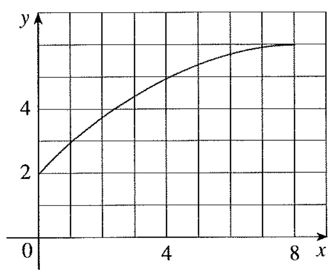
5
$$d = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_{i-1})\,\Delta t = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i)\,\Delta t$$

We will see in Section 4.4 that this is indeed true.

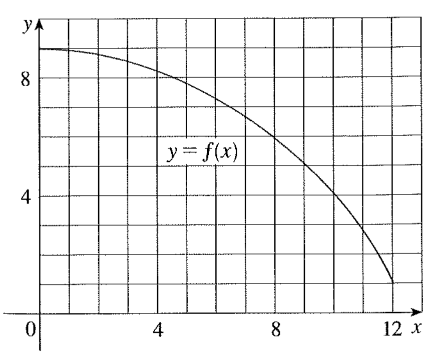
Because Equation 5 has the same form as our expressions for area in Equations 2 and 3, it follows that the distance traveled is equal to the area under the graph of the velocity function. In Chapter 5 we will see that other quantities of interest in the natural and social sciences—such as the work done by a variable force or the cardiac output of the heart—can also be interpreted as the area under a curve. So when we compute areas in this chapter, bear in mind that they can be interpreted in a variety of practical ways.

4.1 Exercises

1. (a) By reading values from the given graph of f, use four rectangles to find a lower estimate and an upper estimate for the area under the given graph of f from $x = 0$ to $x = 8$. In each case sketch the rectangles that you use.
(b) Find new estimates using eight rectangles in each case.



2. (a) Use six rectangles to find estimates of each type for the area under the given graph of f from $x = 0$ to $x = 12$.
  (i) L_6 (sample points are left endpoints)
  (ii) R_6 (sample points are right endpoints)
  (iii) M_6 (sample points are midpoints)
(b) Is L_6 an underestimate or overestimate of the true area?
(c) Is R_6 an underestimate or overestimate of the true area?
(d) Which of the numbers L_6, R_6, or M_6 gives the best estimate? Explain.



3. (a) Estimate the area under the graph of $f(x) = \cos x$ from $x = 0$ to $x = \pi/2$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?
(b) Repeat part (a) using left endpoints.

4. (a) Estimate the area under the graph of $f(x) = \sqrt{x}$ from $x = 0$ to $x = 4$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?
(b) Repeat part (a) using left endpoints.

5. (a) Estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$ using three rectangles and right endpoints. Then improve your estimate by using six rectangles. Sketch the curve and the approximating rectangles.
(b) Repeat part (a) using left endpoints.
(c) Repeat part (a) using midpoints.
(d) From your sketches in parts (a)–(c), which appears to be the best estimate?

6. (a) Graph the function

$$f(x) = 1/(1 + x^2) \qquad -2 \leq x \leq 2$$

(b) Estimate the area under the graph of f using four approximating rectangles and taking the sample points to be (i) right endpoints and (ii) midpoints. In each case sketch the curve and the rectangles.
(c) Improve your estimates in part (b) by using eight rectangles.

7. Evaluate the upper and lower sums for $f(x) = 2 + \sin x$, $0 \leq x \leq \pi$, with $n = 2, 4$, and 8. Illustrate with diagrams like Figure 14.

8. Evaluate the upper and lower sums for $f(x) = 1 + x^2$, $-1 \leq x \leq 1$, with $n = 3$ and 4. Illustrate with diagrams like Figure 14.

9–10 With a programmable calculator (or a computer), it is possible to evaluate the expressions for the sums of areas of approximating rectangles, even for large values of n, using looping. (On a TI use the Is> command or a For-EndFor loop, on a Casio use Isz, on an HP or in BASIC use a FOR-NEXT loop.) Compute the sum of the areas of approximating rectangles using equal subintervals and right endpoints for $n = 10, 30, 50$, and 100. Then guess the value of the exact area.

9. The region under $y = x^4$ from 0 to 1

10. The region under $y = \cos x$ from 0 to $\pi/2$

CAS **11.** Some computer algebra systems have commands that will draw approximating rectangles and evaluate the sums of their areas, at least if x_i^* is a left or right endpoint. (For instance, in Maple use `leftbox`, `rightbox`, `leftsum`, and `rightsum`.)

(a) If $f(x) = 1/(x^2 + 1)$, $0 \leqslant x \leqslant 1$, find the left and right sums for $n = 10, 30$, and 50.

(b) Illustrate by graphing the rectangles in part (a).

(c) Show that the exact area under f lies between 0.780 and 0.791.

CAS **12.** (a) If $f(x) = x/(x + 2)$, $1 \leqslant x \leqslant 4$, use the commands discussed in Exercise 11 to find the left and right sums for $n = 10, 30$, and 50.

(b) Illustrate by graphing the rectangles in part (a).

(c) Show that the exact area under f lies between 1.603 and 1.624.

13. The speed of a runner increased steadily during the first three seconds of a race. Her speed at half-second intervals is given in the table. Find lower and upper estimates for the distance that she traveled during these three seconds.

t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0
v (m/s)	0	1.9	3.3	4.5	5.5	5.9	6.2

14. Speedometer readings for a motorcycle at 12-second intervals are given in the table.

(a) Estimate the distance traveled by the motorcycle during this time period using the velocities at the beginning of the time intervals.

(b) Give another estimate using the velocities at the end of the time periods.

(c) Are your estimates in parts (a) and (b) upper and lower estimates? Explain.

t (s)	0	12	24	36	48	60
v (m/s)	9.1	8.5	7.6	6.7	7.3	8.2

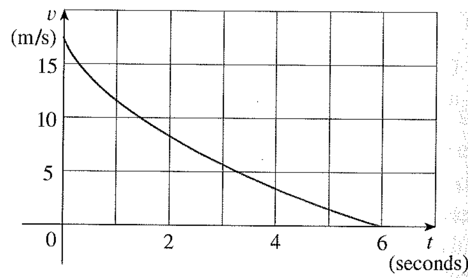
15. Oil leaked from a tank at a rate of $r(t)$ liters per hour. The rate decreased as time passed and values of the rate at two-hour time intervals are shown in the table. Find lower and upper estimates for the total amount of oil that leaked out.

t (h)	0	2	4	6	8	10
$r(t)$ (L/h)	8.7	7.6	6.8	6.2	5.7	5.3

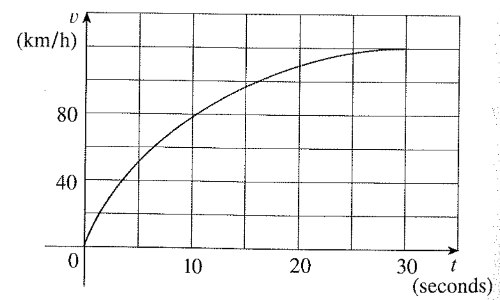
16. When we estimate distances from velocity data, it is sometimes necessary to use times $t_0, t_1, t_2, t_3, \ldots$ that are not equally spaced. We can still estimate distances using the time periods $\Delta t_i = t_i - t_{i-1}$. For example, on May 7, 1992, the space shuttle *Endeavour* was launched on mission STS-49, the purpose of which was to install a new perigee kick motor in an Intelsat communications satellite. The table, provided by NASA, gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters. Use these data to estimate the height above the earth's surface of the *Endeavour*, 62 seconds after liftoff.

Event	Time (s)	Velocity (m/s)
Launch	0	0
Begin roll maneuver	10	56
End roll maneuver	15	97
Throttle to 89%	20	136
Throttle to 67%	32	226
Throttle to 104%	59	404
Maximum dynamic pressure	62	440
Solid rocket booster separation	125	1265

17. The velocity graph of a braking car is shown. Use it to estimate the distance traveled by the car while the brakes are applied.



18. The velocity graph of a car accelerating from rest to a speed of 120 km/h over a period of 30 seconds is shown. Estimate the distance traveled during this period.



19–21 Use Definition 2 to find an expression for the area under the graph of f as a limit. Do not evaluate the limit.

19. $f(x) = \dfrac{2x}{x^2 + 1}, \quad 1 \leqslant x \leqslant 3$

20. $f(x) = x^2 + \sqrt{1 + 2x}, \quad 4 \leqslant x \leqslant 7$

21. $f(x) = \sqrt{\sin x}, \quad 0 \leqslant x \leqslant \pi$

22–23 Determine a region whose area is equal to the given limit. Do not evaluate the limit.

22. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(5 + \frac{2i}{n}\right)^{10}$

23. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n}$

24. (a) Use Definition 2 to find an expression for the area under the curve $y = x^3$ from 0 to 1 as a limit.

(b) The following formula for the sum of the cubes of the first n integers is proved in Appendix E. Use it to evaluate the limit in part (a).

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

25. Let A be the area under the graph of an increasing continuous function f from a to b, and let L_n and R_n be the approximations to A with n subintervals using left and right endpoints, respectively.

(a) How are A, L_n, and R_n related?

(b) Show that

$$R_n - L_n = \frac{b-a}{n}[f(b) - f(a)]$$

Then draw a diagram to illustrate this equation by showing that the n rectangles representing $R_n - L_n$ can be reassembled to form a single rectangle whose area is the right side of the equation.

(c) Deduce that

$$R_n - A < \frac{b-a}{n}[f(b) - f(a)]$$

26. If A is the area under the curve $y = \sin x$ from 0 to $\pi/2$, use Exercise 25 to find a value of n such that $R_n - A < 0.0001$.

CAS **27.** (a) Express the area under the curve $y = x^5$ from 0 to 2 as a limit.

(b) Use a computer algebra system to find the sum in your expression from part (a).

(c) Evaluate the limit in part (a).

CAS **28.** (a) Express the area under the curve $y = x^4 + 5x^2 + x$ from 2 to 7 as a limit.

(b) Use a computer algebra system to evaluate the sum in part (a).

(c) Use a computer algebra system to find the exact area by evaluating the limit of the expression in part (b).

CAS **29.** Find the exact area under the cosine curve $y = \cos x$ from $x = 0$ to $x = b$, where $0 \leqslant b \leqslant \pi/2$. (Use a computer algebra system both to evaluate the sum and compute the limit.) In particular, what is the area if $b = \pi/2$?

30. (a) Let A_n be the area of a polygon with n equal sides inscribed in a circle with radius r. By dividing the polygon into n congruent triangles with central angle $2\pi/n$, show that

$$A_n = \tfrac{1}{2}nr^2 \sin\left(\frac{2\pi}{n}\right)$$

(b) Show that $\lim_{n\to\infty} A_n = \pi r^2$. [*Hint:* Use Equation 2.4.2 on page 141.]

4.2 The Definite Integral

We saw in Section 4.1 that a limit of the form

$$\boxed{1} \qquad \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

arises when we compute an area. We also saw that it arises when we try to find the distance traveled by an object. It turns out that this same type of limit occurs in a wide variety of situations even when f is not necessarily a positive function. In Chapters 5 and 8 we will see that limits of the form $\boxed{1}$ also arise in finding lengths of curves, volumes of solids, centers of mass, force due to water pressure, and work, as well as other quantities. We therefore give this type of limit a special name and notation.