In the following example we estimate the rate of change of the national debt with respect to time. Here the function is defined not by a formula but by a table of values.

t	$D(t)$
1994	414.0
1996	469.5
1998	467.3
2000	456.4
2002	442.3

EXAMPLE 7 Let $D(t)$ be the Canadian gross public debt at time t. The table in the margin gives approximate values of this function by providing midyear estimates, in billions of dollars, from 1994 to 2002. Interpret and estimate the value of $D'(1998)$.

SOLUTION The derivative $D'(1998)$ means the rate of change of D with respect to t when $t = 1998$, that is, the rate of increase of the national debt in 1998.

According to Equation 5,

$$D'(1998) = \lim_{t \to 1998} \frac{D(t) - D(1998)}{t - 1998}$$

t	$\dfrac{D(t) - D(1998)}{t - 1998}$
1994	13.3
1996	−1.1
2000	−5.5
2002	−6.3

So we compute and tabulate values of the difference quotient (the average rates of change) as shown in the table at the left. From this table we see that $D'(1998)$ lies somewhere between -1.1 and -5.5 billion dollars per year. [Here we are making the reasonable assumption that the debt didn't fluctuate wildly between 1998 and 2002.] We estimate that the rate of change of the Canadian debt in 1998 was the average of these two numbers, namely

$$D'(1998) \approx -3.3 \text{ billion dollars per year}$$

The minus sign means that the debt was *decreasing* at that time.

Another method would be to plot the debt function and estimate the slope of the tangent line when $t = 1998$.

A Note on Units

The units for the average rate of change $\Delta D/\Delta t$ are the units for ΔD divided by the units for Δt, namely, billions of dollars per year. The instantaneous rate of change is the limit of the average rates of change, so it is measured in the same units: billions of dollars per year.

In Examples 3, 6, and 7 we saw three specific examples of rates of change: the velocity of an object is the rate of change of displacement with respect to time; marginal cost is the rate of change of production cost with respect to the number of items produced; the rate of change of the debt with respect to time is of interest in economics. Here is a small sample of other rates of change: In physics, the rate of change of work with respect to time is called *power*. Chemists who study a chemical reaction are interested in the rate of change in the concentration of a reactant with respect to time (called the *rate of reaction*). A biologist is interested in the rate of change of the population of a colony of bacteria with respect to time. In fact, the computation of rates of change is important in all of the natural sciences, in engineering, and even in the social sciences. Further examples will be given in Section 2.7.

All these rates of change are derivatives and can therefore be interpreted as slopes of tangents. This gives added significance to the solution of the tangent problem. Whenever we solve a problem involving tangent lines, we are not just solving a problem in geometry. We are also implicitly solving a great variety of problems involving rates of change in science and engineering.

2.1 Exercises

1. A curve has equation $y = f(x)$.
(a) Write an expression for the slope of the secant line through the points $P(3, f(3))$ and $Q(x, f(x))$.
(b) Write an expression for the slope of the tangent line at P.

2. Graph the curve $y = \sin x$ in the viewing rectangles $[-2, 2]$ by $[-2, 2]$, $[-1, 1]$ by $[-1, 1]$, and $[-0.5, 0.5]$ by $[-0.5, 0.5]$. What do you notice about the curve as you zoom in toward the origin?

3. (a) Find the slope of the tangent line to the parabola $y = 4x - x^2$ at the point $(1, 3)$
(i) using Definition 1 (ii) using Equation 2
(b) Find an equation of the tangent line in part (a).

Graphing calculator or computer required **1.** Homework Hints available at stewartcalculus.com

(c) Graph the parabola and the tangent line. As a check on your work, zoom in toward the point (1, 3) until the parabola and the tangent line are indistinguishable.

4. (a) Find the slope of the tangent line to the curve $y = x - x^3$ at the point (1, 0)
(i) using Definition 1 (ii) using Equation 2
(b) Find an equation of the tangent line in part (a).
(c) Graph the curve and the tangent line in successively smaller viewing rectangles centered at (1, 0) until the curve and the line appear to coincide.

5–8 Find an equation of the tangent line to the curve at the given point.

5. $y = 4x - 3x^2$, $(2, -4)$ **6.** $y = x^3 - 3x + 1$, $(2, 3)$

7. $y = \sqrt{x}$, $(1, 1)$ **8.** $y = \dfrac{2x + 1}{x + 2}$, $(1, 1)$

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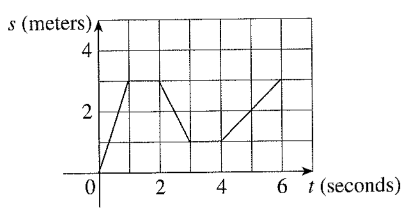
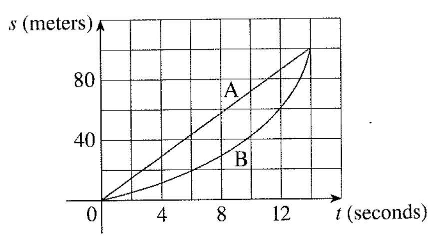
9. (a) Find the slope of the tangent to the curve $y = 3 + 4x^2 - 2x^3$ at the point where $x = a$.
(b) Find equations of the tangent lines at the points (1, 5) and (2, 3).
(c) Graph the curve and both tangents on a common screen.

10. (a) Find the slope of the tangent to the curve $y = 1/\sqrt{x}$ at the point where $x = a$.
(b) Find equations of the tangent lines at the points (1, 1) and $(4, \frac{1}{2})$.
(c) Graph the curve and both tangents on a common screen.

11. (a) A particle starts by moving to the right along a horizontal line; the graph of its position function is shown. When is the particle moving to the right? Moving to the left? Standing still?
(b) Draw a graph of the velocity function.



12. Shown are graphs of the position functions of two runners, A and B, who run a 100-m race and finish in a tie.



(a) Describe and compare how the runners run the race.
(b) At what time is the distance between the runners the greatest?
(c) At what time do they have the same velocity?

13. If a ball is thrown into the air with a velocity of 10 m/s, its height (in meters) after t seconds is given by $y = 10t - 4.9t^2$. Find the velocity when $t = 2$.

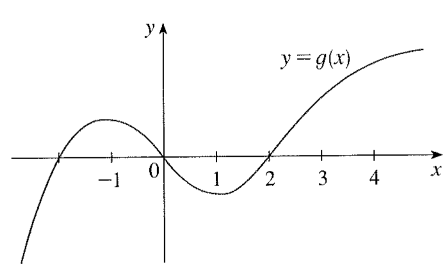
14. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in meters) after t seconds is given by $H = 10t - 1.86t^2$.
(a) Find the velocity of the rock after one second.
(b) Find the velocity of the rock when $t = a$.
(c) When will the rock hit the surface?
(d) With what velocity will the rock hit the surface?

15. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s = 1/t^2$, where t is measured in seconds. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$, and $t = 3$.

16. The displacement (in meters) of a particle moving in a straight line is given by $s = t^2 - 8t + 18$, where t is measured in seconds.
(a) Find the average velocity over each time interval:
(i) [3, 4] (ii) [3.5, 4]
(iii) [4, 5] (iv) [4, 4.5]
(b) Find the instantaneous velocity when $t = 4$.
(c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a) and the tangent line whose slope is the instantaneous velocity in part (b).

17. For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

$0 \qquad g'(-2) \qquad g'(0) \qquad g'(2) \qquad g'(4)$



18. Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.

19. If an equation of the tangent line to the curve $y = f(x)$ at the point where $a = 2$ is $y = 4x - 5$, find $f(2)$ and $f'(2)$.

20. If the tangent line to $y = f(x)$ at (4, 3) passes through the point (0, 2), find $f(4)$ and $f'(4)$.

21. Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 3$, $f'(1) = 0$, and $f'(2) = -1$.

22. Sketch the graph of a function g for which $g(0) = g(2) = g(4) = 0$, $g'(1) = g'(3) = 0$, $g'(0) = g'(4) = 1$, $g'(2) = -1$, $\lim_{x \to 5^-} g(x) = \infty$, and $\lim_{x \to -1^+} g(x) = -\infty$.

23. If $f(x) = 3x^2 - x^3$, find $f'(1)$ and use it to find an equation of the tangent line to the curve $y = 3x^2 - x^3$ at the point $(1, 2)$.

24. If $g(x) = x^4 - 2$, find $g'(1)$ and use it to find an equation of the tangent line to the curve $y = x^4 - 2$ at the point $(1, -1)$.

25. (a) If $F(x) = 5x/(1 + x^2)$, find $F'(2)$ and use it to find an equation of the tangent line to the curve $y = 5x/(1 + x^2)$ at the point $(2, 2)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

26. (a) If $G(x) = 4x^2 - x^3$, find $G'(a)$ and use it to find equations of the tangent lines to the curve $y = 4x^2 - x^3$ at the points $(2, 8)$ and $(3, 9)$.
(b) Illustrate part (a) by graphing the curve and the tangent lines on the same screen.

27–32 Find $f'(a)$.

27. $f(x) = 3x^2 - 4x + 1$

28. $f(t) = 2t^3 + t$

29. $f(t) = \dfrac{2t + 1}{t + 3}$

30. $f(x) = x^{-2}$

31. $f(x) = \sqrt{1 - 2x}$

32. $f(x) = \dfrac{4}{\sqrt{1 - x}}$

33–38 Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

33. $\displaystyle\lim_{h \to 0} \frac{(1 + h)^{10} - 1}{h}$

34. $\displaystyle\lim_{h \to 0} \frac{\sqrt[4]{16 + h} - 2}{h}$

35. $\displaystyle\lim_{x \to 5} \frac{2^x - 32}{x - 5}$

36. $\displaystyle\lim_{x \to \pi/4} \frac{\tan x - 1}{x - \pi/4}$

37. $\displaystyle\lim_{h \to 0} \frac{\cos(\pi + h) + 1}{h}$

38. $\displaystyle\lim_{t \to 1} \frac{t^4 + t - 2}{t - 1}$

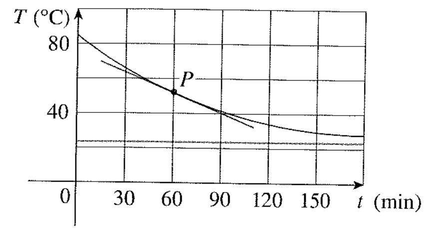
39–40 A particle moves along a straight line with equation of motion $s = f(t)$, where s is measured in meters and t in seconds. Find the velocity and the speed when $t = 5$.

39. $f(t) = 100 + 50t - 4.9t^2$

40. $f(t) = t^{-1} - t$

41. A warm can of soda is placed in a cold refrigerator. Sketch the graph of the temperature of the soda as a function of time. Is the initial rate of change of temperature greater or less than the rate of change after an hour?

42. A roast turkey is taken from an oven when its temperature has reached 85°C and is placed on a table in a room where the temperature is 24°C. The graph shows how the temperature of the turkey decreases and eventually approaches room temperature. By measuring the slope of the tangent, estimate the rate of change of the temperature after an hour.



43. The table shows the number of passengers P that arrived in Ireland by air, in millions.

Year	2001	2003	2005	2007	2009
P	8.49	9.65	11.78	14.54	12.84

(a) Find the average rate of increase of P
 (i) from 2001 to 2005 (ii) from 2003 to 2005
 (iii) from 2005 to 2007
 In each case, include the units.
(b) Estimate the instantaneous rate of growth in 2005 by taking the average of two average rates of change. What are its units?

44. The number N of locations of a popular coffeehouse chain is given in the table. (The numbers of locations as of October 1 are given.)

Year	2004	2005	2006	2007	2008
N	8569	10,241	12,440	15,011	16,680

(a) Find the average rate of growth
 (i) from 2006 to 2008 (ii) from 2006 to 2007
 (iii) from 2005 to 2006
 In each case, include the units.
(b) Estimate the instantaneous rate of growth in 2006 by taking the average of two average rates of change. What are its units?
(c) Estimate the instantaneous rate of growth in 2006 by measuring the slope of a tangent.
(d) Estimate the intantaneous rate of growth in 2007 and compare it with the growth rate in 2006. What do you conclude?

45. The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.
(a) Find the average rate of change of C with respect to x when the production level is changed
 (i) from $x = 100$ to $x = 105$
 (ii) from $x = 100$ to $x = 101$
(b) Find the instantaneous rate of change of C with respect to x when $x = 100$. (This is called the *marginal cost*. Its significance will be explained in Section 2.7.)

46. If a cylindrical tank holds 100,000 liters of water, which can be drained from the bottom of the tank in an hour, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V(t) = 100{,}000\left(1 - \tfrac{1}{60}t\right)^2 \qquad 0 \leqslant t \leqslant 60$$

Find the rate at which the water is flowing out of the tank (the instantaneous rate of change of V with respect to t) as a function of t. What are its units? For times $t = 0$, 10, 20, 30, 40, 50, and 60 min, find the flow rate and the amount of water remaining in the tank. Summarize your findings in a sentence or two. At what time is the flow rate the greatest? The least?

47. The cost of producing x kilograms of gold from a new gold mine is $C = f(x)$ dollars.
(a) What is the meaning of the derivative $f'(x)$? What are its units?
(b) What does the statement $f'(50) = 36$ mean?
(c) Do you think the values of $f'(x)$ will increase or decrease in the short term? What about the long term? Explain.

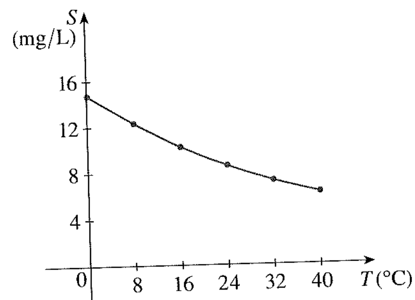
48. The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$.
(a) What is the meaning of the derivative $f'(5)$? What are its units?
(b) Suppose there is an unlimited amount of space and nutrients for the bacteria. Which do you think is larger, $f'(5)$ or $f'(10)$? If the supply of nutrients is limited, would that affect your conclusion? Explain.

49. Let $T(t)$ be the temperature (in °C) in Manila t hours after noon on July 19, 2011. The table shows values of this function recorded every two hours. What is the meaning of $T'(5)$? Estimate its value.

t	1	3	5	7	9	11
T	32	32	31	27	26	25

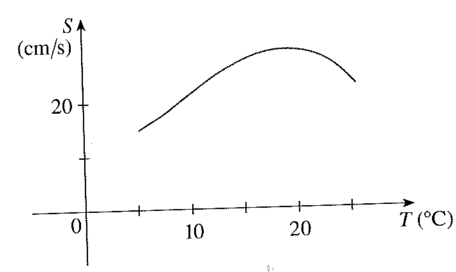
50. The quantity (in kilograms) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per kilogram is $Q = f(p)$.
(a) What is the meaning of the derivative $f'(8)$? What are its units?
(b) Is $f'(8)$ positive or negative? Explain.

51. The quantity of oxygen that can dissolve in water depends on the temperature of the water. (So thermal pollution influences the oxygen content of water.) The graph shows how oxygen solubility S varies as a function of the water temperature T.
(a) What is the meaning of the derivative $S'(T)$? What are its units?
(b) Estimate the value of $S'(16)$ and interpret it.



Adapted from Kupchella & Hyland, *Environmental Science: Living Within the System of Nature*, 2d ed.; © 1989. Printed and electronically reproduced by permission of Pearson Education, Inc., Upper Saddle River, NJ.

52. The graph shows the influence of the temperature T on the maximum sustainable swimming speed S of Coho salmon.
(a) What is the meaning of the derivative $S'(T)$? What are its units?
(b) Estimate the values of $S'(15)$ and $S'(25)$ and interpret them.



53–54 Determine whether $f'(0)$ exists.

53. $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

54. $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$