# M E T U – N C C
## Mathematics Group

| | Calculus with Analytic Geometry<br>Final Exam | |
|---|---|---|
| Code : MAT 119<br>Acad.Year : 2010-2011<br>Semester : Spring<br>Instructors : A.D./H.T./B.W.<br>Date : 09.06.2011<br>Time : 13.00<br>Duration : 120 minutes | Last Name :<br>Name : KEY   Stud. No :<br>Dept. :   Sec. No :<br>Signature :<br>**6** Questions on **8** Pages<br>Total **100** Points | |

| 1 (18) | 2 (24) | 3 (12) | 4 (15) | 5 (21) | 6 (10) | | |
|---|---|---|---|---|---|---|---|
| | | | | | | | |

**Q.1 (6 × 3 = 18 pts)** Evaluate the following limits:

(a) $\lim\limits_{x\to 0} \dfrac{\int_x^0 (e^t + t - 1)\,dt}{x^2} = \left\{\dfrac{0}{0}\right\} \stackrel{L.R.}{=} -\lim\limits_{x\to 0} \dfrac{\frac{d}{dx}\int_0^x (e^t+t-1)\,dt}{2x} =$

$= -\lim\limits_{x\to 0} \dfrac{e^x + x - 1}{2x} = \left\{\dfrac{0}{0}\right\} \stackrel{L.R.}{=} -\lim\limits_{x\to 0} \dfrac{e^x+1}{2} =$

$= \boxed{-1}$

(b) $\lim\limits_{x\to\infty} (1+e^{-x})^x = \{1^\infty\}$. Put $y = y(x) = (1+e^{-x})^x$.

Then $\ln(y) = x\ln(1+e^{-x}) = \dfrac{\ln(1+e^{-x})}{\frac{1}{x}}$, and

$\lim\limits_{x\to\infty} \ln(y) = \left\{\dfrac{0}{0}\right\} \stackrel{L.R.}{=} \lim\limits_{x\to\infty} \dfrac{-\frac{e^{-x}}{1+e^{-x}}}{-\frac{1}{x^2}} = \lim\limits_{x\to\infty} \dfrac{x^2}{1+e^x}$

$= \left\{\dfrac{\infty}{\infty}\right\} = \lim\limits_{x\to\infty} \dfrac{2x}{e^x} = \left\{\dfrac{\infty}{\infty}\right\} = \lim\limits_{x\to\infty} \dfrac{2}{e^x} = 0.$

Based on continuity of exp-function, we derive that

$\lim\limits_{x\to\infty} y(x) = \lim\limits_{x\to\infty} e^{\ln(y)} = e^0 = \boxed{1}$

(c) $\lim_{x\to 0} \frac{\cos x^2}{x^2}$ Note that

$$\frac{\cos(x^2)}{x^2} \geqslant \frac{1}{2x^2} \quad \text{for small } x.$$

$$\boxed{\infty}$$

(d) $\lim_{x\to 0} \frac{\sin x^2 - x^2}{x^6} = \left\{\frac{0}{0}\right\} \overset{L.R.}{=} \lim_{x\to 0} \frac{2x\cos(x^2) - 2x}{6x^5} =$

$$= \lim_{x\to 0} \frac{\cos(x^2) - 1}{3x^4} = \left\{\frac{0}{0}\right\} \overset{L.R}{=} \lim_{x\to 0} \frac{-2x\sin(x^2)}{12x^3} =$$

$$= -\frac{1}{6} \lim_{x\to 0} \frac{\sin(x^2)}{x^2} = \boxed{-\frac{1}{6}}$$

(e) $\lim_{x\to\infty} (\ln x)^{1/x} = \{\infty^0\}$ Put $y = y(x) = (\ln(x))^{1/x}$. Then

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{\ln(\ln(x))}{x} = \left\{\frac{\infty}{\infty}\right\} \overset{L.R.}{=} \lim_{x\to\infty} \frac{1}{\ln(x)\cdot x} = 0$$

As above, $\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln(y)} = e^0 = \boxed{1.}$

(f) $\lim_{x\to 0^+} x^{\csc x} = \{0^\infty\}$. If $y = x^{\csc(x)}$ then

$$\ln(y) = \frac{\ln(x)}{\sin(x)} \text{ and } \lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} \ln(x) \frac{1}{\sin(x)} = -\infty$$

Hence $\lim_{x\to 0^+} y(x) = \lim_{x\to 0^+} e^{\ln(y)} = \boxed{0}$

**Last Name:** ______________________ **Name:** ______________

**Q.2 (6 × 4 = 24 pts)** Evaluate the following integrals:

(a) $\int \frac{\sqrt{x+1}}{x-3}\,dx = \left| \begin{matrix} u=\sqrt{x+1}, & u^2-1=x \\ du=\frac{dx}{2u} & u^2-4=x-3 \end{matrix} \right| = 2\int \frac{u^2\,du}{u^2-4} =$

$= 2u + 8\int \frac{du}{(u-2)(u+2)} = 2u + 2\left(\int \frac{du}{u-2} - \int \frac{du}{u+2}\right)$

$= 2u + \ln\left(\frac{u-2}{u+2}\right)^2 + C =$

$= \boxed{2\sqrt{x+1} + \ln\left(\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right)^2 + C}$

(b) $\int x^{3/2}\ln(5x)\,dx = \frac{2}{5}\int (x^{5/2})'\ln(5x)\,dx = \frac{2}{5}x^{5/2}\ln(5x)$

$-\frac{2}{5}\int x^{5/2}\frac{1}{5x}5\,dx = \frac{2}{5}x^{5/2}\ln(5x) - \frac{2}{5}\int x^{3/2}dx$

$= \boxed{\frac{2}{5}x^{5/2}\ln(5x) - \frac{2}{5}\cdot\frac{2}{5}x^{5/2} + C}$

(c) $\int \sin^3(2x)\cos^4(2x)\,dx = -\frac{1}{2}\int (1-\cos^2(2x))\cos^4(2x)(-2\sin(2x)dx)$

$= \left| \begin{matrix} u=\cos(2x) \\ du=-2\sin(2x)dx \end{matrix} \right| = -\frac{1}{2}\int (1-u^2)u^4\,du =$

$= -\frac{1}{2}\frac{u^5}{5} + \frac{1}{2}\frac{u^7}{7} + C = \boxed{\frac{\cos^7(2x)}{14} - \frac{\cos^5(2x)}{10} + C}$

(d) $\int \frac{x-8}{x^3+4x}\,\mathrm{d}x = \int \frac{x-8}{x(x^2+4)}\,dx$ , $\frac{x-8}{x(x^2+4)} = \frac{-2}{x} + \frac{2x+1}{x^2+4}$

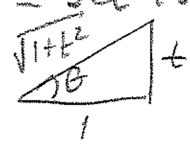
– expansion into the partial fractions.

Then $\int \frac{x-8}{x(x^2+4)}\,dx = -2\int \frac{dx}{x} + \int \frac{2x+1}{x^2+4}\,dx =$

$$= \boxed{-2\ln|x| + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + \ln(x^2+4) + C}$$

(e) $\int \frac{1}{(x^2+4x+5)^{3/2}}\,\mathrm{d}x = \int \frac{dx}{((x+2)^2+1)^{3/2}} = \left| \begin{array}{l} t = x+2 \\ dt = dx \end{array} \right| =$

$$= \int \frac{dt}{(t^2+1)^{3/2}} = \left| \begin{array}{l} t = \tan(\theta),\ -\pi/2 < \theta < \pi/2 \\ dt = \sec^2(\theta)\,d\theta,\ (t^2+1)^{3/2} = \sec^3(\theta) \end{array} \right|$$




$$= \int \frac{d\theta}{\sec(\theta)} = \sin(\theta) + C = \frac{t}{\sqrt{t^2+1}} + C = \boxed{\frac{x+2}{\sqrt{x^2+4x+5}} + C}$$

(f) $\int \frac{x^3-1}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{x^3\,dx}{\sqrt{1-x^2}} - \arcsin(x).$

$$\int \frac{x^3\,dx}{\sqrt{1-x^2}} = \left| \begin{array}{l} u = \sqrt{1-x^2},\ x^2 = 1-u^2 \\ -u\,du = x\,dx \end{array} \right| = -\int \frac{(1-u^2)u\,du}{u}$$

$$= \int (u^2-1)\,du = \frac{u^3}{3} - u + C = \frac{(1-x^2)^{3/2}}{3} - \sqrt{1-x^2} + C$$

$$\boxed{\frac{(1-x^2)^{3/2}}{3} - \sqrt{1-x^2} - \arcsin(x) + C}$$

**Q.3 (3 × 4 = 12 pts)** Determine whether the following integrals converge or diverge. If they converge, what do they converge to?

**(a)** $\displaystyle\int_0^4 \frac{x}{x^2-4}\,\mathrm{d}x$

If $2 < x \leq 4$ then $\dfrac{x}{x^2-4} = \dfrac{x}{(x-2)(x+2)}$

$\geqslant \dfrac{2}{(x-2)6} = \dfrac{1}{3(x-2)}$. But $\displaystyle\int_2^4 \frac{dx}{3(x-2)}$ diverges,

whence so is $\displaystyle\int_2^4 \frac{x\,dx}{x^2-4}$ (In particular, $\displaystyle\int_0^4 \frac{x\,dx}{x^2-4}$)

$\boxed{\text{Diverges}}$

**(b)** $\displaystyle\int_4^\infty \frac{1}{\sqrt{x}e^{\sqrt{x}}}\,\mathrm{d}x = \lim_{b\to\infty} \int_4^b \frac{dx}{\sqrt{x}\,e^{\sqrt{x}}} = 2\lim_{b\to\infty}\int_2^{\sqrt{b}} \frac{du}{e^u}$

$\displaystyle = -2\lim_{b\to\infty}\left(e^{-u}\Big|_2^{\sqrt{b}}\right) = \boxed{\frac{2}{e^2} \quad \text{(Converges)}}$

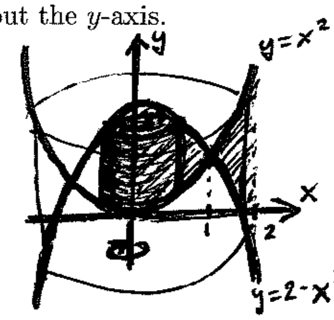
**(c)** $\displaystyle\int_1^\infty \frac{\sin^4 x + 1}{x^{1/4}}\,\mathrm{d}x$ For all $x \geqslant 1$, we have

$\dfrac{\sin^4(x)+1}{x^{1/4}} \geqslant \dfrac{1}{x^{1/4}}$. But $\displaystyle\int_1^\infty \frac{dx}{x^{1/4}}$ diverges

thanks to the p-Test. Hence the original one diverges too. $\boxed{\text{Diverges}}$

**Q.4** (7 + 8 = 15 pts) Consider the region between the curves $y = x^2$ and $y = 2 - x^2$ from $x = 0$ to $x = 2$.

(a) Find the volume of the solid obtained by rotating this region about the $y$-axis.




(Cylindrical Shells)

Point of intersection:

$$x^2 = 2 - x^2$$
$$2x^2 = 2$$
$$x = \pm 1$$

Volume:

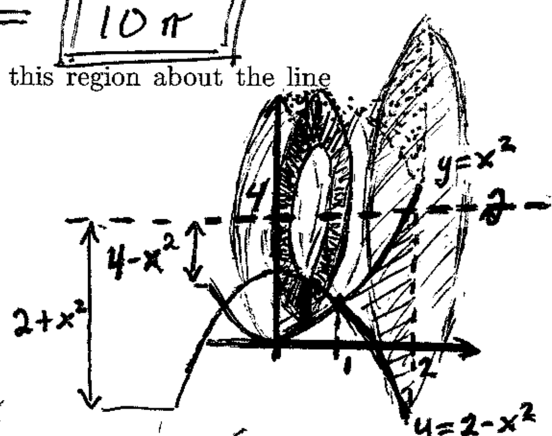
$$V = \int_0^1 2\pi x(2 - x^2 - x^2)\,dx + \int_1^2 2\pi x\,(x^2 - (2 - x^2))\,dx$$

$$= \int_0^1 4\pi\,(x - x^3)\,dx + \int_1^2 4\pi\,(x^3 - x)\,dx$$

$$= 4\pi\left(\tfrac{1}{2} - \tfrac{1}{4}\right) + 4\pi\left(\left(\tfrac{16}{4} - \tfrac{4}{2}\right) - \left(\tfrac{1}{4} - \tfrac{1}{2}\right)\right)$$

$$= 4\pi\left(\tfrac{1}{4} + 2 + \tfrac{1}{4}\right) = \boxed{10\pi}$$

(b) Find the volume of the solid obtained by rotating this region about the line $y = 4$.




(Disks/Washers)

Volume:

$$V = \int_0^1 \pi\left((4 - x^2)^2 - (2 + x^2)^2\right)dx$$
$$+ \int_1^2 \pi\left((2 + x^2)^2 - (4 - x^2)^2\right)dx$$

$$= \pi \int_0^1 16 - 8x^2 + x^4 - 4 - 4x^2 - x^4\,dx$$

$$+ \pi \int_1^2 4 + 4x^2 + x^4 - 16 + 8x^2 - x^4\,dx$$

$$= \pi \int_0^1 12 - 12x^2\,dx + \pi \int_1^2 -12 + 12x^2\,dx$$

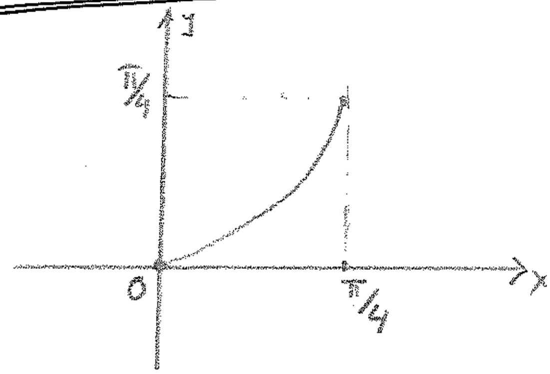
$$= \pi\left(12 - \tfrac{12}{3}\right) + \pi\left(-24 + \tfrac{12}{3}\cdot 8 + 12 - \tfrac{12}{3}\right)$$

$$= \pi(8) + \pi(16) = \boxed{24\pi}$$

**Q.5 (3 × 7 = 21 pts)** The following parts involve the curve $y = x\tan x$ from $x = 0$ to $x = \pi/4$.

(a) Write, but **do NOT** evaluate, the integral which gives the **arclength** of $y = x\tan x$ from $x = 0$ to $x = \pi/4$.

$$\boxed{\ell = \int_0^{\pi/4} \sqrt{1 + \left(\tan(x) + x\sec^2(x)\right)^2}\, dx}$$




(b) Write, but **do NOT** evaluate, the integral which gives the **surface area** of the surface obtained by rotating this curve about the $y$-axis.

$$\boxed{S = \int_0^{\pi/4} 2\pi x \sqrt{1 + \left(\tan(x) + x\sec^2(x)\right)^2}\, dx}$$

(c) Write, but **do NOT** evaluate, the integral which gives the **surface area** of the surface obtained by rotating this curve about $y = -1$.

$$\boxed{S = \int_0^{\pi/4} 2\pi\left(1 + x\tan(x)\right) \sqrt{1 + \left(\tan(x) + x\sec^2(x)\right)^2}\, dx}$$

**Q.6 (10 pts)** Use the mean value theorem (MVT) to show that $\ln x < x - 1$ for $x > 1$.

**NOTE.** This problem was included also in the 2nd Midterm Exam, which is supposed to be somewhat comrehensive.

Based on MVT, we obtain that

$$\frac{\ln(x) - \ln(1)}{x-1} = \ln'(x)\Big|_{x=c}$$

for a certain $c \in (1, x)$. But

$$\ln'(x)\Big|_{x=c} = \frac{1}{c} < 1.$$ Hence

$$\frac{\ln(x)}{x-1} < 1 \quad \text{or} \quad \ln(x) < x-1$$