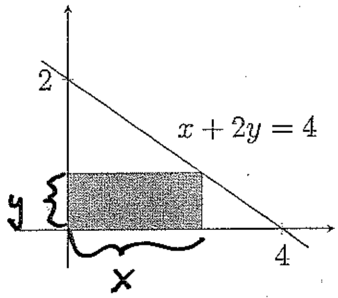
	Calculus and Analytical Geometry Final	
Code : *Math 119* Acad.Year: *2010-2011* Semester : *Fall* Date : *10.1.2011* Time : *9:30* Duration : *180 minutes*	Last Name: Name : Department: Signature:	Student No.: Section:
	6 QUESTIONS ON 8 PAGES TOTAL 100 POINTS	

1	2	3	4	5	6		

Please show your work in all questions.

1. (12 points) Find the maximum area of a rectangle in the first quadrant with two sides along the axes and the fourth vertex on the line $x + 2y = 4$.



$$x = 4 - 2y$$

2 pts

$$\text{Area} = x \cdot y = (4-2y)y$$
$$= 4y - 2y^2$$

6 pts

$$0 = (\text{Area})' = 4 - 4y$$
$$4y = 4$$

2 pts

$$y = 1 \Rightarrow x = 2$$

2 pts

$$\text{Area} = 1 \cdot 2 = \boxed{2}$$

$(\text{Area})' = 4 - 4y$
$(\text{Area})'' = -4 < 0$
So critical point is max

2. (5+5+5=15 points) (a) If $y = x^{\cos x} \ln x$, find y'.

$$y = x^{\cos x} \ln x$$

$$\ln y = \ln\left(x^{\cos x} \ln x\right)$$

$$\ln y = \cos x \ln x + \ln \ln x$$

$$\frac{y'}{y} = \frac{\cos x}{x} - \sin x \ln x + \frac{1}{x \ln x}$$

$$y' = \left(\frac{\cos x}{x} - \sin x \ln x + \frac{1}{x \ln x}\right) x^{\cos x} \ln x$$

$$\left(\text{Note: } y' = \left((\cos x) x^{\cos x - 1} - \sin x \ln x \, x^{\cos x}\right) \ln x + x^{\cos x} \cdot \frac{1}{x}\right)$$

(b) Let $y = \dfrac{(x+1)(x+2)(x+3)(x+4)}{(x+5)(x+6)(x+7)(x+8)}$. Find y'.

$$\ln y = \ln(x+1) + \ln(x+2) + \ln(x+3) + \ln(x+4) - \ln(x+5) - \ln(x+6) - \ln(x+7) - \ln(x+8)$$

$$\frac{y'}{y} = \left(\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \frac{1}{x+4} - \frac{1}{x+5} - \frac{1}{x+6} - \frac{1}{x+7} - \frac{1}{x+8}\right)$$

$$y' = \left(\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \frac{1}{x+4} - \frac{1}{x+5} - \frac{1}{x+6} - \frac{1}{x+7} - \frac{1}{x+8}\right)\left(\frac{(x+1)(x+2)(x+3)(x+4)}{(x+5)(x+6)(x+7)(x+8)}\right)$$

(c) Suppose that $x^3 + xy^2 + x^2y + y^3 = 10$. Find y' in terms of x and y.

$$\frac{d}{dx}$$

$$3x^2 + (y^2 + 2xyy') + (2xy + x^2y') + 3y^2y' = 0$$

$$(x^2 + 3y^2 + 2xy)y' = -(3x^2 + y^2 + 2xy)$$

$$y' = -\frac{3x^2 + y^2 + 2xy}{x^2 + 3y^2 + 2xy}$$

3. (4+4+4+4=16 points) Evaluate the following limits:

(a) $\lim_{x\to\infty} e^{-x}\cos x$. $= \lim_{x\to\infty} \frac{\cos x}{e^x}$ Since $-1 \leq \cos x \leq 1$ then $\frac{-1}{e^x} \leq \frac{\cos x}{e^x} \leq \frac{1}{e^x}$

Using Squeeze Theorem: $\lim_{x\to\infty}\left(\frac{-1}{e^x}\right) \leq \lim_{x\to\infty}\left(\frac{\cos x}{e^x}\right) \leq \lim_{x\to\infty}\left(\frac{1}{e^x}\right)$

We have: $0 \leq \lim_{x\to\infty} \frac{\cos x}{e^x} \leq 0$

then $\lim_{x\to\infty} \frac{\cos x}{e^x} = 0$ //

(b) $\lim_{x\to 0^+} (\sin x)^{\tan x}$. This is an indeterminate of type 0^0.

Then $\lim_{x\to 0^+} (\sin x)^{\tan x} = e^{\left[\lim_{x\to 0^+} \tan x \cdot \ln\sin x\right]} = e^{\left[\lim_{x\to 0^+} \frac{\ln\sin x}{\cot x}\right]}$

The power is an indeterminate of the type ∞/∞.

Using L'Hospital: $e^{\left[\lim_{x\to 0^+} \frac{\cos x\cdot\frac{1}{\sin x}}{-1/\sin^2 x}\right]} = e^{\left[\lim_{x\to 0^+} -\cos x\cdot\sin x\right]}$

$= e^0 = 1$ //

(c) $\lim_{x\to 2} \frac{x^x - 4}{x-2}$. This is an indeterminate of type $0/0$.

Using L'Hospital: $\lim_{x\to 2} \frac{(x^x-4)'}{(x-2)'} = \lim_{x\to 2} \frac{(x^x)'}{1}$

For derivative of $y = x^x$, $\ln y = x\ln x$ $\quad y'\cdot\frac{1}{y} = \ln x + x\cdot\frac{1}{x}$

$y' = x^x\cdot(1+\ln x)$

So, our limit becomes: $\lim_{x\to 2} x^x(1+\ln x) = 2^2\,(1+\ln 2)$ //

(d) $\lim_{x\to\infty} \frac{\sinh x}{\cosh x}$. This is an indeterminate form of type ∞/∞.

By sinh and cosh definition:

$\lim_{x\to\infty} \frac{\sinh x}{\cosh x} = \lim_{x\to\infty} \frac{(e^x - e^{-x})/2}{(e^x+e^{-x})/2} = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$= \lim_{x\to\infty} \frac{e^x(1-e^{-2x})}{e^x(1+e^{2x})} = \lim_{x\to\infty} \frac{1-0}{1+0} = 1$ //

4. (5+5+5+5+5+5=30 points) Evaluate the following integrals (or write "divergent"):

(a) $\displaystyle\int_0^{\pi/4} \tan^3 x \sec^2 x \, dx.$

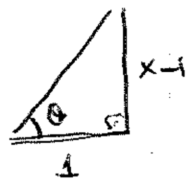
Use substitution $u = \tan x$ then,

$$\int_0^1 u^3 \, du = \frac{u^4}{4}\Big|_0^1 = \frac{1}{4}$$

(b) $\displaystyle\int \frac{x^3 - 2x^2 + 4x}{x^2 - 2x + 2} \, dx.$ First, make the division. Then, integral becomes

$$\int x + \frac{2x}{(x-1)^2+1} \, dx = \int x \, dx + \int \frac{\overbrace{2x-2 \, dx}^{du}}{\underbrace{(x-1)^2+1}_{u}} + \int \frac{2\,dx}{(x-1)^2+1}$$

$x-1 = \tan\theta$



$$= \int x \, dx + \int \frac{du}{u} + \int \frac{2 \cancel{\sec^2\theta} \, d\theta}{\cancel{\sec^2\theta}}$$

$$= \frac{x^2}{2} + \ln|(x-1)^2+1| + 2.\tan^{-1}(x-1) + C$$

(c) $\displaystyle\int_0^{\pi/2} \frac{\sin t \cos t}{\sqrt{1+\sin^2 t}} \, dt.$

Use substitution $\sin t = \tan\theta$ then, $\cos t \, dt = \sec^2\theta \, d\theta$

$$\int_0^{\pi/4} \frac{\tan\theta . \sec^{\cancel{2}}\theta \, d\theta}{\cancel{\sec\theta}} = \sec\theta \Big|_0^{\pi/4} = \sqrt{2} - 1$$

(d) $\int_0^{\infty} \frac{\sqrt{\arctan x}}{x^2+1}\,dx.$

Use substitution $u=\tan^{-1}x$, then $x=\tan u$, $dx=\sec^2 u\,du$

Then, $$\int_0^{\pi/2} \frac{\sqrt{u}\,\cancel{\sec^2 u}\,du}{\cancel{\sec^2 u}} = \frac{u^{3/2}}{3/2}\Big|_0^{\pi/2} = \frac{2}{3}\left(\frac{\pi}{2}\right)^{3/2}.$$

(e) $\int \frac{e^x+2}{e^{2x}+1}\,dx.$ Use substitution $u=e^x$, $du=e^x\,dx \Rightarrow \frac{du}{u}=dx$.

Then, $\int \frac{u+2}{u^2+1}\cdot\frac{du}{u} = \int \frac{A}{u}+\frac{Bu+C}{u^2+1}\,du$ where $A(u^2+1)+(Bu+C)u=u+2$

$\Rightarrow A=2,\ B=-2;\ C=1.$

$$=\int \frac{2}{u}-\frac{2u-1}{u^2+1}\,du$$

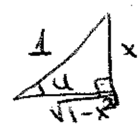
$$=\int \frac{2}{u}\,du-\int\frac{2u\,du}{u^2+1}+\int\frac{1}{u^2+1}\,du = 2\ln|u|-\ln|u^2+1|+\tan^{-1}u+C$$

$$=2.\ln|e^x|-\ln|e^{2x}+1|+\tan^{-1}(e^x)+C.$$

(f) $\int \frac{8\arcsin x}{x^3}\,dx.$ Use substitution $u=\arcsin x \Rightarrow x=\sin u,\ dx=\cos u\,du$

$$\int 8.u.\overbrace{\frac{\cos u\,du}{\sin^3 u}}^{=dv} \xrightarrow[\text{by parts}]{\text{using integration}} 8.\left[u.\left(\frac{1}{-2\sin^2 u}\right)-\int -\frac{1}{2\sin^2 u}\,du\right]$$

$$=-\frac{4u}{\sin^2 u}-4\cot u+C$$



$$=-\frac{4.\arcsin x}{x^2}-4.\frac{\sqrt{1-x^2}}{x}+C$$

5)

a)

$$\text{Arclength} = \int_0^2 \sqrt{1 + \left(2^{-x^2/2} \cdot \ln 2 \cdot (x)\right)^2} \, dx$$

$$\text{Volume} = \int_0^2 \pi \left(2^{-x^2/2}\right)^2 \, dx$$

$$\text{Surface Area} = \int_0^2 2\pi \, e^{-x^2/2} \sqrt{1 + \left(2^{-x^2/2} \ln 2 \, (-x)\right)^2} \, dx$$

b) $\Delta x = \frac{2-0}{4} = 0.5$

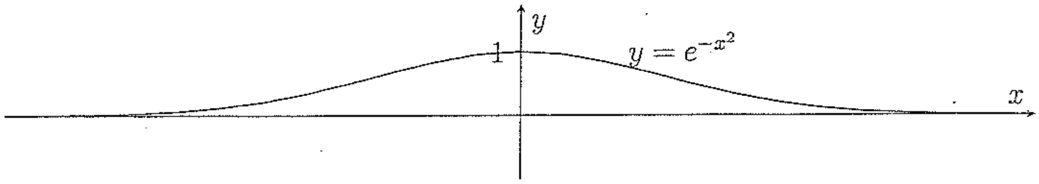
Left end point: $R_4 = 0.5\left(\pi \cdot 2^0 + \pi\, 2^{-1/4} + \pi\, 2^{-1} + \pi\, 2^{-9/4}\right)$

Right endpoint $R_4 = 0.5\left(\pi\, 2^{-1/4} + \pi\, 2^{-1} + \pi\, 2^{-9/4} + \pi\, 2^{-4}\right)$

Since $\pi\left(2^{-x^2/2}\right)^2$ is decreasing on $[0,2]$ the volume can't be more than the left end point approximation. Since the left end point approximation is less than 3π, the volume is also less than 3π.

6. (1+1+2+3+2+2+3=14 points) This question first gives some preliminary information, then asks you to sketch a graph.

Recall that the graph of the Gaussian function $f(x) = e^{-x^2}$ has the following shape:

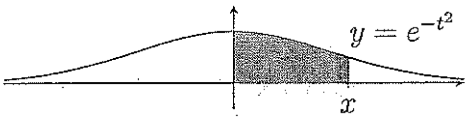


The Gaussian function is very important for several applications, especially in probability theory and statistics. The total area under $f(x)$ can be calculated (with methods from the course MAT 120), and one finds: $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

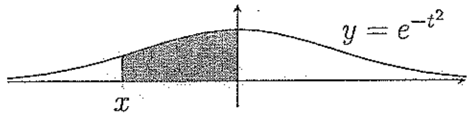
This question asks you to sketch the graph of the function

$$F(x) = \int_0^x e^{-t^2} dt.$$

Note that one has the following interpretation for $F(x)$ in terms of areas:



$F(x) =$ Area for $x > 0$



$F(x) = -$Area for $x < 0$

1 (a) Find the domain of $F(x)$.

$\mathrm{dom}(F(x)) = \mathbb{R}$

(b) Find the intercepts.

1 $F(x) = 0 \iff x = 0$. So, $(0,0)$ all intercepts

(c) Determine the symmetries of the function.

2 $F(-x) = \int_0^{-x} e^{-t^2} dt = \left| \begin{array}{l} u = -t \\ du = -dt \\ t=0 \Rightarrow u=0 \\ t=-x \Rightarrow u=x. \end{array} \right| = -\int_0^x e^{-u^2} du = -F(x).$

So, $F(x)$ is an odd function. The graph is sysmetric about the origin

(d) Find all asymptotes. It has only horizontal asymptotes.

$$\lim_{x \to +\infty} F(x) = \lim_{x \to +\infty} \int_0^x e^{-t^2} dt = \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}, \text{ for}$$

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-t^2} dt = \int_{-\infty}^{0} e^{-t^2} dt + \int_0^\infty e^{-t^2} dt = 2\int_0^\infty e^{-t^2} dt \quad (y = e^{-t^2} \text{ is even function})$$

(e) Find the intervals of increase / decrease, and find the critical points.

$F'(x) = e^{-x^2} > 0$ always increasing

(f) Discuss the concavity and find the inflection points.

$F''(x) = -2x\, e^{-x^2} \Rightarrow (F''(x) = 0 \Leftrightarrow x = 0)$ Moreover,

	$x<0$	$x>0$
$F''(x)$	+	−
$F(x)$	conc. up	conc. down.

(g) Sketch the graph of $y = F(x)$.