SOLUTIONS

# METU - NCC

| | Precalculus<br>Final | |
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| Code : *Math 100*<br>Acad.Year: *2011-2012*<br>Semester : *Spring*<br>Date : *4.6.2012*<br>Time : *9:00*<br>Duration : *120 minutes* | Last Name:<br>Name :<br>Department:<br>Signature:<br>12 QUESTIONS ON 4 PAGES<br>TOTAL 100 POINTS | Student No.:<br>Section: |

| 1 (8) | 2 (6) | 3 (8) | 4 (8) | 5 (8) | 6 (8) | 7 (8) | 8 (10) | 9 (10) | 10 (8) | 11 (10) | 12 (8) | |
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**No calculators!** No course notes! Please write your answers in the boxes provided.

**1.** *(8 pts)* If $f(x) = ka^x$ is an exponential function such that

$$f(1) = 34, \quad f(0) = 17, \quad f(-2) = \frac{17}{4}$$

find the values of $k$ and $a$.

$f(1) = ka^1 = k.a = 34$

$f(0) = k.a^0 = k = 17$

$\Rightarrow a = 2$

$\boxed{k = 17 \qquad a = 2}$

**2.** *(6 pts)* An exponentially growing population triples in 15 years. How long does it take this population to double?

$A(t) = A_0 . 2^{t/d}$

$A(15) = A_0 . 2^{15/d} = 3. A_0$

$\Rightarrow 2^{15/d} = 3 \Rightarrow \frac{15}{d} = \log_2 3 \Rightarrow d = \frac{15}{\log_2 3}$ . We need: $A(t) = A_0 . 2^{t.\log_2 3/15} = 2.A_0$

$\Rightarrow t.\log_2 3 = 15 \Rightarrow t = 15.\log_3 2$

$\boxed{15 . \log_3 2 \text{ years}}$

**3.** *(8 pts)* The following parts are about angles.

a) Find the equivalents of the angles below in $[0, 2\pi)$

$\frac{34\pi}{4} = 8\pi + \frac{2\pi}{4} = 8\pi + \frac{\pi}{2}$

$\frac{-23\pi}{6} = -\left(4\pi - \frac{\pi}{6}\right) = -4\pi + \frac{\pi}{6}$

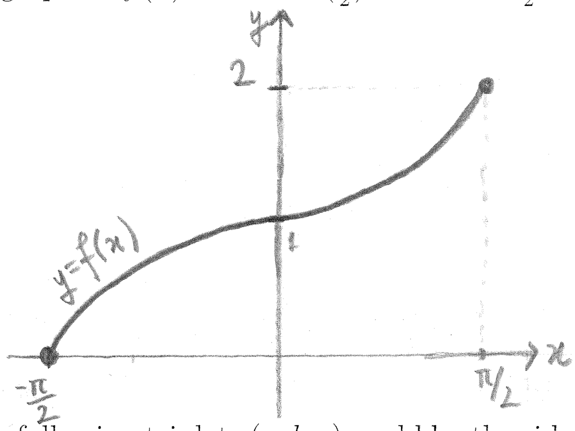
$\boxed{\frac{34\pi}{4} \equiv \frac{\pi}{2} \qquad \frac{-23\pi}{6} \equiv \frac{\pi}{6}}$

b) Convert the following from degrees to radians.

$18 . \frac{\pi}{180} = \frac{\pi}{10}$ | $216 . \frac{\pi}{180} = \frac{6\pi}{5}$

$\boxed{18^\circ = \frac{\pi}{10} \qquad 216^\circ = \frac{6\pi}{5}}$

4. *(8 pts)* Sketch the graph of $f(x) = 1 + \tan(\frac{x}{2})$ from $x = \frac{-\pi}{2}$ to $x = \frac{\pi}{2}$.




Note:

$f\left(-\frac{\pi}{2}\right) = 1 + \tan\left(-\frac{\pi}{4}\right)$

$= 1 - \tan\left(\frac{\pi}{4}\right) = 0.$

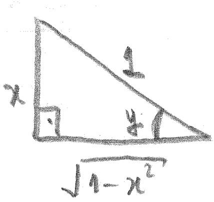
$f(0) = 1.$

$f\left(\frac{\pi}{2}\right) = 1 + \tan\left(\frac{\pi}{4}\right) = 2$

5. *(8 pts)* Which of the following triplets $(a, b, c)$ could be the side lengths of a **right** triangle?

- (1,2,3) No! $(1^2 + 2^2 \neq 3^2)$
- $(3, 4, 5)$ Yes. $(3^2 + 4^2 = 5^2)$
- $(12, 5, 13)$ Yes. $(12^2 + 5^2 = 13^2)$
- $(1, \sqrt{2}, \sqrt{2})$ No! $(1^2 + \sqrt{2}^2 \neq \sqrt{2}^2)$
- $(\sqrt{12}, 3\sqrt{3}, 8)$ No! $(\sqrt{12}^2 + (3\sqrt{3})^2 \neq 8^2)$

6. *(8 pts)* Write $\cos(\arcsin(x))$ in terms of x.




Say $\arcsin(x) = y$

So:

$\cos(\arcsin(x)) = \sqrt{1-x^2}$

$\boxed{\cos(\arcsin(x)) = \sqrt{1-x^2}}$

7. *(8 pts)* Find the value of $\sin(\frac{13\pi}{12})$.

$\sin\left(\frac{13\pi}{12}\right) = \sin\left(\frac{\pi}{3} + \frac{3\pi}{4}\right)$

$= \sin\left(\frac{\pi}{3}\right) \cdot \cos\left(\frac{3\pi}{4}\right) + \sin\left(\frac{3\pi}{4}\right) \cdot \cos\left(\frac{\pi}{3}\right)$

$= \sin\left(\frac{\pi}{3}\right) \cdot -\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) \cdot \cos\left(\frac{\pi}{3}\right)$

$= -\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{2} - \sqrt{6}}{4}$

$\boxed{\sin\left(\frac{13\pi}{12}\right) = \frac{\sqrt{2} - \sqrt{6}}{4}}$

8. *(10 pts)* If $\sin(x) + \cos(x) = a$ then write the $\sin(2x)$ in terms of $a$.

$$\boxed{\sin(2x) = a^2 - 1}$$

$$\sin(x) + \cos(x) = a$$

$$\Leftrightarrow \underbrace{\sin^2(x) + \cos^2(x)}_{1} + \underbrace{2\sin(x)\cos(x)}_{\sin(2x)} = a^2$$

$$\Leftrightarrow 1 + \sin(2x) = a^2$$

$$\Leftrightarrow \sin(2x) = a^2 - 1$$

9. *(10 pts)* Use basic trigonometric identities to verify the following:

$$\frac{\cot(x) - \tan(x)}{2} = \frac{1}{\tan(2x)}$$

(left hand side) $\text{LHS} = \dfrac{\frac{1}{\tan(x)} - \tan(x)}{2}$

$$= \frac{1 - \tan^2(x)}{2\tan(x)} = \frac{1}{\frac{2\tan(x)}{1-\tan^2(x)}} = \frac{1}{\tan(2x)} = \text{RHS}$$ (right hand side)

10. *(8 pts)* Compute $\cos(75°)$ exactly.

$$\boxed{\cos(75°) = \frac{\sqrt{6} - \sqrt{2}}{4}}$$

$$\cos(75°) = \cos\left(\frac{5\pi}{12}\right) = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$$

$$= \cos\left(\frac{\pi}{4}\right)\cdot\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{4}\right)\cdot\sin\left(\frac{\pi}{6}\right)$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

OR: $\sin\left(\frac{13\pi}{12}\right) = \sin\left(\pi + \frac{\pi}{12}\right) = -\sin\left(\frac{\pi}{12}\right) = -\cos\left(\frac{\pi}{2} - \frac{\pi}{12}\right)$

$$= -\cos\left(\frac{5\pi}{12}\right)$$

see Question 7.

$$= -\frac{\sqrt{2} - \sqrt{6}}{4}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

**11.** *(10 pts)* Find all values of $x$ which make the following true:

$$\sec^2(x) = 2.$$

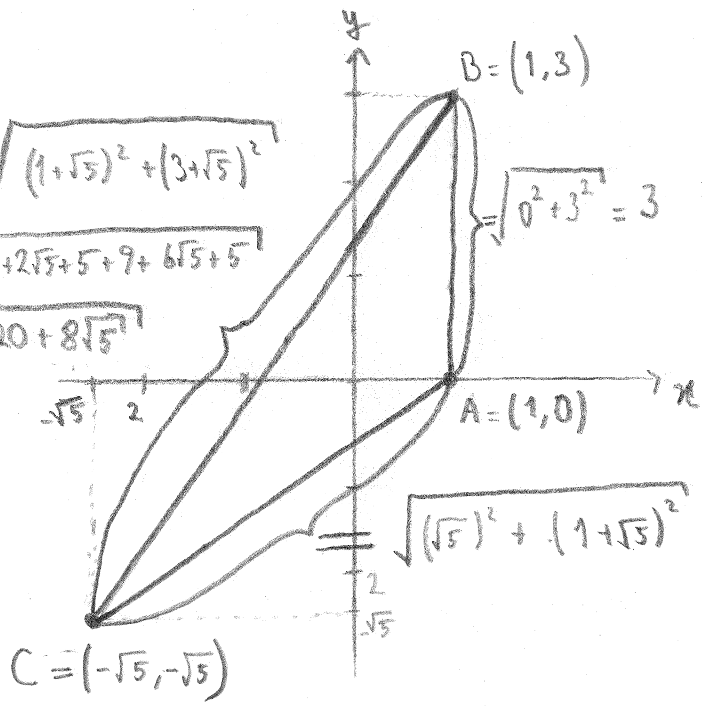
$$\sec^2(x) = 2 \Leftrightarrow \frac{1}{\cos^2(x)} = 2 \Leftrightarrow \cos(x) = \mp\frac{\sqrt{2}}{2}$$

$$\cos(x) = \frac{\sqrt{2}}{2} \quad \text{OR} \quad \cos(x) = -\frac{\sqrt{2}}{2}$$

$$\Rightarrow \left( \begin{array}{c} x = \frac{\pi}{4} + 2k\pi;\ k \in \mathbb{Z} \\ \text{OR} \\ x = \left(2\pi - \frac{\pi}{4}\right) + 2k\pi;\ k\in\mathbb{Z} \end{array} \right) \quad \Rightarrow \left( \begin{array}{c} x = \pi + \frac{\pi}{4} + 2k\pi,\ k\in\mathbb{Z} \\ \text{OR} \\ x = \pi - \frac{\pi}{4} + 2k\pi,\ k\in\mathbb{Z} \end{array} \right)$$

$$\boxed{x = \begin{array}{l} \frac{\pi}{4} + 2\pi k;\ k\in\mathbb{Z} \ \text{OR} \\ \frac{7\pi}{4} + 2\pi k;\ k\in\mathbb{Z} \ \text{OR} \\ \frac{3\pi}{4} + 2\pi k;\ k\in\mathbb{Z} \ \text{OR} \\ \frac{5\pi}{4} + 2\pi k;\ k\in\mathbb{Z} \end{array}}$$

**12.** *(8 pts)* A triangle has vertices $A = (1, 0)$, $B = (1, 3)$, $C = (-\sqrt{5}, -\sqrt{5})$. Compute the lengths of all edges and the measure of the angle $\angle BAC$.




$$\sqrt{(1+\sqrt{5})^2 + (3+\sqrt{5})^2} = \sqrt{1 + 2\sqrt{5} + 5 + 9 + 6\sqrt{5} + 5} = \sqrt{20 + 8\sqrt{5}}$$

$$= \sqrt{0^2 + 3^2} = 3$$

$$\sqrt{(\sqrt{5})^2 + (1+\sqrt{5})^2} = \sqrt{5 + 1 + 2\sqrt{5} + 5} = \sqrt{11 + 2\sqrt{5}}$$

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| length$(AB) = 3$ |
| length$(BC) = \sqrt{20 + 8\sqrt{5}}$ |
| length$(AC) = \sqrt{11 + 2\sqrt{5}}$ |
| $\angle BAC = \arccos\left(-\frac{\sqrt{5}}{\sqrt{11+2\sqrt{5}}}\right)$ |

To find $\angle BAC = \alpha$:

$$|BC|^2 = |BA|^2 + |AC|^2 - 2|BA| \cdot |AC| \cdot \cos(\alpha)$$

i.e.: $$\cos(\alpha) = -\frac{|BC|^2 - (|BA|^2 + |AC|^2)}{2|BA| \cdot |AC|}$$

So: $$\cos(\alpha) = -\frac{20 + 8\sqrt{5} - 9 - 11 - 2\sqrt{5}}{2 \cdot 3 \cdot \sqrt{11 + 2\sqrt{5}}} = -\frac{6\sqrt{5}}{6\sqrt{11+2\sqrt{5}}} = -\frac{\sqrt{5}}{\sqrt{11+2\sqrt{5}}}$$

$$\Rightarrow \alpha = \arccos\left(-\frac{\sqrt{5}}{[illegible]}\right)$$