# METU - NCC

| | PRECALCULUS FINAL EXAM | |
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| Code : *MAT 100*<br>Acad.Year: *2011-2012*<br>Semester : *Fall*<br>Date : *13.1.2012*<br>Time : *9:00*<br>Duration : *120 minutes* | Last Name:<br>Name :<br>Department:<br>Signature:<br>Student No.:<br>Section:<br>12 QUESTIONS ON 4 PAGES<br>TOTAL 100 POINTS | |

| 1 (8) | 2 (6) | 3 (8) | 4 (8) | 5 (10) | 6 (7) | 7 (7) | 8 (7) | 9 (10) | 10 (7) | 11 (12) | 12 (10) | |
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No calculators! No course notes! Please write your answers in the boxes provided.

1. *(4+4pts)* If $f(x) = ka^x$ is an exponential function such that

$$f(-1) = 40, \quad f(0) = 20, \quad f(2) = 5$$

find the values of $k$ and $a$.

$f(0) = k\,a^0 = 20 \Rightarrow k = 20$

$f(-1) = k\,a^{-1} = 40 \Rightarrow a = \frac{1}{2}$

$\boxed{k = 20 \qquad a = \frac{1}{2}}$

2. *(6pts)* An exponentially growing population doubles in 10 years. How long does it take this population to triple?

$P(x) = k a^x$

$P(0) = k \;,\; P(10) = k\,a^{10} = 2k$

So, $a^{10} = 2 \;,\; \ln a = \frac{\ln 2}{10}$

$P(t) = k a^t = 3k \Rightarrow a^t = 3 \Rightarrow t = \frac{\ln 3}{\ln a} \Rightarrow t = \frac{\ln 3}{\frac{\ln 2}{10}} = 10\,\frac{\ln 3}{\ln 2}$

$\boxed{t = 10 \ln 3/\ln 2}$

3. *(4×2pts)* The following parts are about angles.

a) Find the equivalents of the angles below in $[0, 2\pi)$

$\frac{17}{4} = 4 + \frac{1}{4}$

$\frac{-13}{6} = -4 + \frac{11}{6}$

$\boxed{\frac{17\pi}{4} \equiv \frac{\pi}{4} \qquad \frac{-13\pi}{6} \equiv \frac{11}{6}\pi}$

b) Convert the following from degrees to radians.

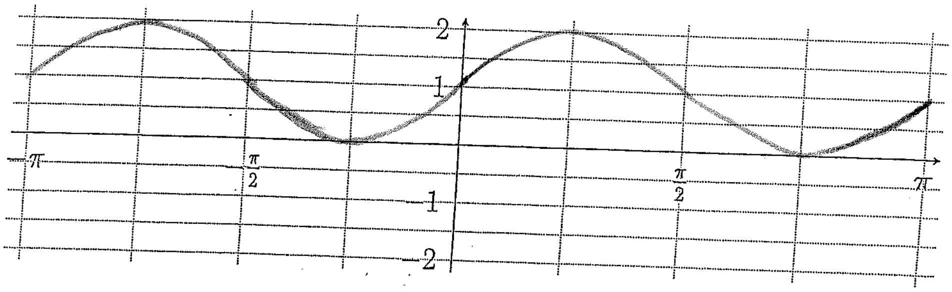
$360^\circ = 2\pi$

So, $36^\circ = \frac{36}{360} \cdot 2\pi = \frac{\pi}{5}$

$195^\circ = \frac{195}{360} \cdot 2\pi = \frac{39\pi}{36}$

$\boxed{36^\circ = \frac{\pi}{5} \qquad 195^\circ = \frac{13\pi}{12}}$

4. *(8pts)* In the plot below, sketch the graph of $f(x) = 1 + \sin(2x)$ from $x = -\pi$ to $x = \pi$.




5. *(5×2pts)* Which of the following triplets $(a, b, c)$ could be the side lengths of a **right** triangle?

(A) $(1, 2, 3)$ $\quad 3^2 \neq 2^2 + 1^2$

(B) $(3, 4, 5)$ $\quad 5^2 = 4^2 + 3^2$

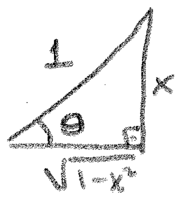
(C) $(12, 5, 13)$ $\quad 13^2 = 12^2 + 5^2$

(D) $(1, \sqrt{2}, \sqrt{2})$ $\quad (\sqrt{2})^2 \neq (\sqrt{2})^2 + 1^2$

(E) $(\sqrt{12}, 3\sqrt{3}, 8)$ $\quad 8^2 \neq (3\sqrt{3})^2 + (\sqrt{12})^2$

6. *(7pts)* Write $\tan(\arcsin(x))$ in terms of x.

$\arcsin x = \theta$




$$\boxed{\tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}}}$$

$$\tan\theta = \frac{x}{\sqrt{1-x^2}}$$

7. *(7pts)* Compute the exact value of $\sin\left(\frac{7\pi}{12}\right)$. *(Your answer may involve square roots.)*
*Hint:* $\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$

$$\boxed{\sin\left(\frac{7\pi}{12}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}}$$

$$\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

8. *(7pts)* If $\sin(x) + \cos(x) = a$ then write the $\sin(2x)$ in terms of $a$.

$$a^2 = \underbrace{\sin^2(x) + \cos^2(x)} + \underbrace{2\sin(x)\cos(x)}$$

$$a^2 = 1 + \sin(2x)$$

$\sin(2x) = a^2 - 1$

So, $\sin(2x) = a^2 - 1$

9. *(10pts)* Use basic trigonometric identities to verify the following:

$$\frac{\tan(x) + \cot(x)}{2} = \frac{1}{\sin(2x)}$$

$$\frac{\tan(x) + \cot(x)}{2} = \frac{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}}{2} = \frac{\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}}{2}$$

$$= \frac{\overbrace{\sin^2 x + \cos^2 x}^{1}}{\underbrace{2\sin x \cos x}_{\sin 2x}}$$

$$= \frac{1}{\sin 2x}$$

10. *(7pts)* Compute the exact value of $\cos(75°)\cos(15°)$.
*(Your answer may involve square roots.)*

$\cos(75°)\cos(15°) = \frac{1}{4}$

$\cos 75° = \sin 15°$

So, $\cos 75° \cos 15° = \sin 15° \cos 15°$

$$= \frac{1}{2} \cdot 2 \sin 15° \cos 15°$$

$$= \frac{1}{2} \cdot \sin 30°$$

$$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

11. *(12pts)* Find all values of $x$ which make the following true:

$$4\cos^2(x) + 8\sin(x) = 7.$$

$\cos^2(x) = 1 - \sin^2(x)$

Eqn becomes,

$4(1-\sin^2(x)) + 8\sin(x) = 7$

$\Rightarrow \quad 4\sin^2(x) - 8\sin(x) + 3 = 0$

$(2\sin(x) - 1)(2\sin x - 3) = 0$

$\sin(x) = \frac{1}{2}$ or $\sin(x) = \frac{3}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ No such $x$!

$$x = \begin{cases} \pi/6 + 2\pi k \\ 5\pi/6 + 2\pi k \end{cases}$$

$k = \ldots, -2, -1, 0, 1, 2, \ldots$

12. *(3×2+4pts)* A triangle has vertices $A = (0,0)$, $B = (1,1)$, $C = (-\sqrt{3}-1, \sqrt{3}-1)$. Compute the lengths of all edges and the measure of the angle $\angle BAC$.

$\text{length}(AB) = \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2}$

$\text{length}(BC) = \sqrt{(-\sqrt{3}-2)^2 + (\sqrt{3}-2)^2} = \sqrt{14}$

$\text{length}(AC) = \sqrt{(-\sqrt{3}-1)^2 + (\sqrt{3}-1)^2} = 2\sqrt{2}$

length$(AB) = \sqrt{2}$

length$(BC) = \sqrt{14}$

length$(AC) = 2\sqrt{2}$

$\angle BAC = 5\pi/6$

Using Cosine Law:

$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB||AC|\cos(B\hat{A}C)$

$14 = 2 + 8 - 2 \cdot \sqrt{2} \cdot 2\sqrt{2} \cdot \cos(B\hat{A}C)$

$\Rightarrow \cos(B\hat{A}C) = -\frac{4}{8} \Rightarrow B\hat{A}C = \frac{5\pi}{6}$