METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES FINAL	
Code : *MAT 120* Acad.Year: *2011-2012* Semester : *Spring* Date : *04.06.2012* Time : *9:00* Duration : *150 minutes*	Last Name: Name : SOLUTIONS Department: Signature: Student No.: Section: 8 QUESTIONS ON 7 PAGES TOTAL 100 POINTS

1 (10)	2 (16)	3 (14)	4 (10)	5 (12)	6 (8)	7 (8)	8 (22)	

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

Good luck!

1. *(7+3 pts)* Let $f(x) = \mathrm{e}^{\cos(x)}$

(a) Find Maclaurin series of the function f up to the x^3 term.

$f(x) = e^{\cos x}$ $\qquad f(0) = e^1$

$f'(x) = -\sin x \, e^{\cos x}$ $\qquad f'(0) = 0$

$f''(x) = (\sin^2 x - \cos x) e^{\cos x}$ $\qquad f''(0) = -e^1$

$f'''(x) = (-\sin^3 x + 3\sin x \cos x + \sin x) e^{\cos x}$ $\qquad f'''(0) = 0$

$$\boxed{f = e + 0 - \underbrace{\frac{e}{2}x^2}_{x^2 \text{ term}} + \underbrace{0}_{x^3 \text{ term}} + \dots}$$

(b) Use the series you obtained in part (a) to approximate $\displaystyle\int_0^{1/2} f(x)\,dx$.

$$\int_0^{1/2} f(x)\,dx \approx \int_0^{1/2} e - \frac{e}{2}x^2\,dx = ex - \frac{e}{6}x^3 \Big|_0^{1/2}$$

$$= \frac{e}{2} - \frac{e}{48}$$

$$= \boxed{\frac{23}{48}e}$$

2. *(4+4+4+4 pts)* Decide whether the following series converge or diverge: (Specify clearly the method used and give all necessary details.)

(a) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$ Integral Test

- $\frac{1}{n\sqrt{\ln n}} \to 0$ and positive
- $\frac{d}{dx}\left(\frac{1}{x\sqrt{\ln x}}\right) = \frac{-\left(\sqrt{\ln x} + \frac{x}{2\sqrt{\ln x}}\right)}{x^2 \ln x} < 0$ so $\frac{1}{n\sqrt{\ln n}}$ decreasing

$$\int_2^{\infty} \frac{(\ln x)^{-1/2}}{x}\,dx = \lim_{R\to\infty} 2(\ln x)^{1/2}\Big|_2^R = \infty$$

$\boxed{\text{Divergent}}$

(b) $\sum_{n=1}^{\infty} \frac{\sqrt{1+\ln(n)}}{n\ln(n)}$ (Hint: Use part (a).)

Limit Comparison Test with $\sum_1 \frac{1}{n\sqrt{\ln n}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{\sqrt{1+\ln n}}{n\ln n}\right)}{\left(\frac{1}{n\sqrt{\ln n}}\right)} = \lim_{n\to\infty} \frac{\sqrt{1+\ln n}}{n\ln n}\cdot n\sqrt{\ln n} = \lim_{n\to\infty} \frac{\sqrt{1+\ln n}}{\sqrt{\ln n}} = 1 \begin{matrix}\neq 0\\ \neq \infty\end{matrix}$$

$\boxed{\text{Divergent}}$ because $\sum_1 \frac{1}{n\sqrt{\ln n}}$ is.

(c) $\sum_{n=1}^{\infty} \frac{n}{n+1}\arctan(n)$

n^{th} Term Test (Test for Divergence)

$$\lim_{n\to\infty} \frac{n}{n+1}\arctan(n) = 1\cdot \pi/2 \neq 0$$

$\boxed{\text{Divergent}}$

(d) $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{(n+1)^3}$

Limit Comparison Test with $\sum_1 \frac{\sqrt{n^3}}{n^3} = \sum_1 \frac{1}{n^{3/2}}$ (convergent p-series)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{\sqrt{n^3+1}}{(n+1)^3}\right)}{\left(\frac{\sqrt{n^3}}{n^3}\right)} = \lim_{n\to\infty} \sqrt{\frac{n^3+1}{n^3}}\cdot\left(\frac{n}{n+1}\right)^3 = 1 \begin{matrix}\neq 0\\ \neq \infty\end{matrix}$$

$\boxed{\text{Convergent}}$ because $\sum_1 1/n^{3/2}$ is.

3. *(5+5+4 pts)* Let $f(x) = \dfrac{2x+2}{2x+3}$

(a) Find the series expansion of f around $c = -1$.

$$f = \frac{2x+2}{2x+3} = 1 - \frac{1}{2x+3} = 1 - \frac{1}{1+2(x-(-1))} = 1 - \frac{1}{1-((-2)(x-(-1)))}$$

$$= 1 - \sum_{n=0}^{\infty} (-2)^n (x-(-1))^n = 1 + \sum_{n=0}^{\infty} (-1)^{n+1} 2^n (x-(-1))^n$$

$n=0$ term is $-2^0 \cdot 1 = -1$

$$= \boxed{\sum_{n=1}^{\infty} (-1)^{n+1} 2^n (x-(-1))^n}$$

(b) Use (a) to write the Taylor series of $\dfrac{1}{(2x+3)^2}$ around $c = -1$.

$$\frac{d}{dx} f = \frac{d}{dx}\left(1 - \frac{1}{2x+3}\right) = \frac{2}{(2x+3)^2} \quad \text{so} \ldots$$

$$\frac{1}{(2x+3)^2} = \frac{1}{2}\frac{d}{dx} f = \frac{1}{2}\frac{d}{dx}\sum_{n=1}^{\infty} (-1)^{n+1} 2^n (x-(-1))^n$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} (-1)^{n+1} 2^n \cdot n\, (x-(-1))^{n-1} = \boxed{\sum_{n=0}^{\infty} (-1)^n 2^n (n+1)(x-(-1))^n}$$

(c) Find the sum $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{3^n}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{3^n} = \sum_{n=1}^{\infty} (-1)^n 2^n (n+1)\left(\frac{1}{6}\right)^n = \sum_{n=1}^{\infty} (-1)^n 2^n (n+1)\left(-\tfrac{5}{6} - (-1)\right)^n$$

missing $n=0$ term

$$= -1 + \sum_{n=0}^{\infty} (-1)^n 2^n (n+1)\left(-\tfrac{5}{6} - (-1)\right)^n$$

$$= -1 + \frac{1}{(2\cdot(-\frac{5}{6})+3)^2} \quad \text{from part (b).}$$

$$= -1 + (3/4)^2$$

$$= \boxed{7/16}$$

4. *(10 pts)* Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{n^{3/2}}{(n^2+2012)120^n}(x-5)^{2n}$

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^{3/2}}{((n+1)^2+2012)120^{n+1}}(x-5)^{2n+2}}{\frac{n^{3/2}}{(n^2+2012)120^n}(x-5)^{2n}}\right|$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{3/2}\left(\frac{n^2+2012}{(n+1)^2+2012}\right)\left(\frac{120^n}{120^{n+1}}\right)\cdot\left|\frac{(x-5)^{2n+2}}{(x-5)^{2n}}\right|$$

$$= 1\cdot 1\cdot\frac{1}{120}\cdot|(x-5)^2|$$

Converges if $\frac{(x-5)^2}{120} < 1$

$(x-5)^2 < 120$

$|x-5| < \sqrt{120}$ ← Radius of convergence.

If $x-5=\sqrt{120}$ then series is

$$\sum_1 \frac{n^{3/2}}{(n^2+2012)\cancel{120^n}}\cdot\cancel{(\sqrt{120})^{2n}}$$

Divergent by limit comparison w/ $\sum_1 \frac{n^{3/2}}{n^2} = \sum_1 \frac{1}{n^{1/2}}$ ← Divergent p-series.

If $x-5=-\sqrt{120}$ then series is

$$\sum_1 \frac{n^{3/2}}{(n^2+2012)\cancel{120^n}}\cancel{(-\sqrt{120})^{2n}}$$

also Divergent

$-\sqrt{120} < x-5 < \sqrt{120}$

$5-\sqrt{120} < x < 5+\sqrt{120}$

Interval of convergence $= \boxed{\left(5-\sqrt{120},\ 5+\sqrt{120}\right)}$

5. *(4+4+4 pts)* Evaluate the following limits

(a) $\lim\limits_{n\longrightarrow\infty}\left(\dfrac{n+2}{n}\right)^n$ $\quad 1^\infty$ Indeterminate Form

$$\lim_{x\to\infty}\left(\frac{x+2}{x}\right)^x = e^{\lim_{x\to\infty} x\ln\left(\frac{x+2}{x}\right)}$$

$$\left(\lim_{x\to\infty} x\ln\left(\frac{x+2}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to\infty}\frac{\left(\frac{x}{x+2}\right)\left(\frac{2}{x^2}\right)}{\frac{1}{x^2}} = 2\right)$$

$$= \boxed{e^2}$$

(b) $\lim\limits_{n\longrightarrow\infty}\dfrac{\sin(\ln(n))}{\ln(n)}$

$$\lim_{n\to\infty}\frac{\sin(\ln n)}{\ln n} = \lim_{m\to\infty}\frac{\sin m}{m}$$

$$-1 \le \sin m \le 1$$

$$\frac{-1}{m} \le \frac{\sin m}{m} \le \frac{1}{m} \qquad \frac{-1}{m} \text{ \& } \frac{1}{m} \to 0$$

So $\lim\limits_{n\to\infty}\dfrac{\sin(\ln(n))}{\ln(n)} = \boxed{0}$ by Squeeze Thm.

(c) $\lim\limits_{n\longrightarrow\infty}\sum\limits_{k=1}^{n}\dfrac{1}{k}$

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{1}{k} = \sum_{k=1}^{\infty}\frac{1}{k}$$ The Harmonic Series!

$$\boxed{\text{Divergent.}}$$

6. *(4+4 pts)* Let R be the solid bounded by the surface $\frac{x^2}{1^2}+\frac{y^2}{2^2}+\frac{z^2}{3^2}=1$

(a) Write (**do not evaluate**) a **double** integral to compute the volume of R.

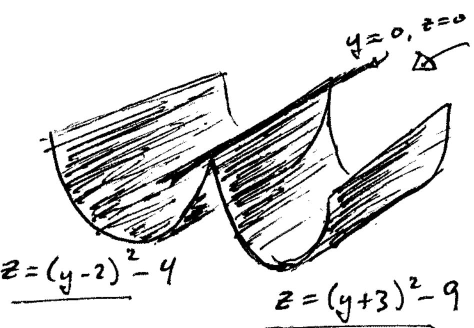
$$\frac{x^2}{1}+\frac{y^2}{4}+\frac{z^2}{9}=1 \rightsquigarrow z=\pm 3\sqrt{1-x^2-y^2/4}$$

$$\text{Volume} = \int_{x=-1}^{x=1}\int_{y=-2\sqrt{1-x^2}}^{y=2\sqrt{1-x^2}} 2\cdot 3\sqrt{1-x^2-y^2/4}\; dy\, dx$$

(b) Write (**do not evaluate**) a **triple** integral to compute the volume of R.

$$\text{Volume} = \int_{x=-1}^{x=1}\int_{y=-2\sqrt{1-x^2}}^{y=2\sqrt{1-x^2}}\int_{z=-3\sqrt{1-x^2-y^2/4}}^{z=3\sqrt{1-x^2-y^2/4}} 1 \; dz\, dy\, dx$$

7. *(8 pts)* Suppose that M and N are two mountains given by the equations $z=(y+3)^2-9$ and $z=(y-2)^2-4$, respectively; where $x \geqslant 0$ and $y \geqslant 0$. Find find the angle of the valley formed by the two mountains.



Mountain ridges are upside-down. Oh well...

Find the angle between gradient vectors:

$z=(y-2)^2-4 \rightsquigarrow \nabla_1 = \langle 0, 2(y-2), -1\rangle$

$z=(y+3)^2-9 \rightsquigarrow \nabla_2 = \langle 0, 2(y+3), -1\rangle$

Intersection line is

$(y-2)^2-4=(y+3)^2-9$

$6y = 2y$

$y=0,\ z=0$

$\longrightarrow$ At $y=0,\ z=0$ these gradients are:

$\nabla_1 = \langle 0, -4, -1\rangle = v_1$

$\nabla_2 = \langle 0, 6, -1\rangle = v_2$

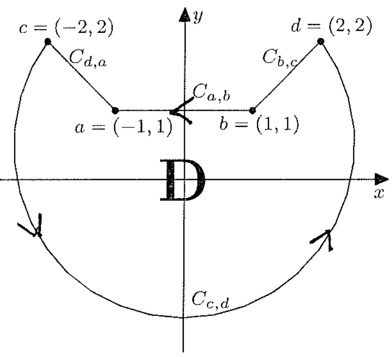
$v_1 \bullet v_2 = |v_1|\,|v_2| \cos\theta$

$-24+1 = \sqrt{37}\cdot\sqrt{17}\cos\theta$

$$\boxed{\theta = \arccos\left(\frac{-23}{\sqrt{37}\sqrt{16}}\right)}$$

8. *(10+5+7 pts)*

Let D be the region lying inside the positively oriented curve shown in the figure, $C_{a,b}$ be the line connecting the point $a = (-1,1)$ to $b = (1,1)$ and $C' = C_{bc} \cup C_{cd} \cup C_{da}$ be the remaining part of the boundary parametrized counterclockwise. Given the vector field $\mathbf{F} = \langle x^3 + y\sin(x), y^3 + x^2y^2 + yx^4 + x^6 \rangle$.



(a) Evaluate $\int_{C_{a,b}} \mathbf{F} \cdot d\mathbf{r}$.

C_{ab} has parameterization $\begin{cases} x = -t \\ y = 1 \end{cases}$ $-1 \le t \le 1$

$\Rightarrow \begin{cases} dx = -dt \\ dy = 0 \end{cases}$

$$\int_{C_{ab}} F \cdot dr = \int_{C_{ab}} (x^3 + y \sin x)\,dx + (y^3 + x^2y^2 + yx^4 + x^6)\,dy$$

$$= \int_{-1}^{1} ((-t)^3 + 1 \cdot \sin(-t))(-dt) + 0 = \frac{t^4}{4} - \cos t \Big|_{-1}^{1} = \boxed{0}$$

(b) Evaluate $\iint_D (2xy^2 + 4x^3y + 6x^5 - \sin(x))\,dA$.

$$\iint_D (2xy^2 + 4x^3y + 6x^5 - \sin(x))\,dx\,dy = \int 0\,dy = \boxed{0}$$

$\|$ function is odd with respect to x
area of integration is symmetric across y-axis

(c) Evaluate $\int_{C'} \mathbf{F} \cdot d\mathbf{r}$.(<u>Hint:</u> Use parts (a) and (b)).

$$\iint_D \frac{d}{dx}(y^3 + x^2y^2 + yx^4 + x^6) - \frac{d}{dy}(x^3 + y \sin x)\,dA = \text{Part (b)} = 0$$

$\|$ Green's Thm.

$$\underbrace{\int_{C_{ab}} F \cdot dr}_{= 0 \text{ by (a)}} + \int_{C'} F \cdot dr$$

So $\boxed{\int_{C'} F \cdot dr = 0}$