a) Compute the determinant

$$\begin{array}{c} \\ R_2-R_1 \\ R_3-R_2 \\ R_4-R_3 \end{array}\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & 5 \\ 1 & 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} =$$

$$= -\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$$

b) Find the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & -3 \\ 0 & 0 & 3 \end{bmatrix}$

using the cofactors.

Since $\det(A) = 6$, we have

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ 3 & 3 & 0 \\ 7 & 3 & 2 \end{bmatrix}^t = \frac{1}{6}\begin{bmatrix} 6 & 3 & 7 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 7/6 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1/3 \end{bmatrix}$$

Q 2

Let $T: \mathbb{R}^4 \to \mathbb{R}^4$, $T(x_1, x_2, x_3, x_4) = (x_1, 2x_1 + 2x_2, 3x_3 + x_4, x_3 + 3x_4)$ be a linear transformation. Find the characteristic polynomial, eigenvalues, eigenspaces of T, and write down a basis f for $\mathbb{R}^4$ such that $M_f^f(T)$ (or $M(T)_f^f$) is diagonal if possible.

Put $A = M_e^e(T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ with the standard basis e for $\mathbb{R}^4$.

Then $\Delta(t) = \det(A - tI) = \begin{vmatrix} 1-t & 0 \\ 2 & 2-t \end{vmatrix} \cdot \begin{vmatrix} 3-t & 1 \\ 1 & 3-t \end{vmatrix} =$

$$= (1-t)(2-t)\left((3-t)^2 - 1\right) = (1-t)(2-t)(2-t)(4-t)$$ 3

$$= (t-2)^2 (t-1)(t-4) \Rightarrow \sigma(T) = \{2^{(2)}, 1^{(1)}, 4^{(1)}\}$$

$\lambda = 2 \Rightarrow A - 2I = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow x_1 = 0,\ x_3 + x_4 = 0$

$V_2 = \text{Span}\{\underbrace{(0,1,0,0)}_{f_1}, \underbrace{(0,0,1,-1)}_{f_2}\}$

2

$\dim(V_2) = 2 \Leftarrow$ algeb. multip.

$\lambda = 1 \Rightarrow A - I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \Rightarrow x_3 = x_4 = 0,\ 2x_1 + x_2 = 0$

2

$V_1 = \text{Span}\,\underbrace{(1,-2,0,0)}_{= f_3}$

$\lambda = 4 \Rightarrow A - 4I = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow x_1 = x_2 = 0,\ x_3 = x_4.$

2

$V_4 = \text{Span}\,\underbrace{(0,0,1,1)}_{f_4}$

Put $f = (f_1, f_2, f_3, f_4)$. Then

4

$$M_f^f(T) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Q3.
Let $T: V \to V$ be a linear transformation over the vector space V.

(a) If $\lambda \in \sigma(T)$ then $\lambda^3 - 1 \in \sigma(T^3 - I)$

Indeed, $\exists\, x \in V \setminus \{0\}$ such that $Tx = \lambda x$. Then

$(T^3 - I)x = T^3 x - x = \lambda^3 x - x = (\lambda^3 - 1)x \Rightarrow \lambda^3 - 1 \in \sigma(T^3 - I)$.

(b) If T is nilpotent then $\sigma(T) = \{0\}$.

Suppose $T^p = 0$. If $Tx = \lambda x$ for $\lambda \in \sigma(T)$, $x \neq 0$, then $0 = T^p x = \lambda^p x \Rightarrow \lambda^p = 0 \Rightarrow \lambda = 0$.

(c) If T is invertible then $\sigma(T^{-1}) = \sigma(T)^{-1}$.

Take $\lambda \in \sigma(T)$. Then $\lambda \neq 0$, for $\ker(T) = \{0\}$.

If $Tx = \lambda x$ then $T(\lambda^{-1} x) = x \Rightarrow T^{-1} T(\lambda^{-1} x) = T^{-1} x$

$\Rightarrow \lambda^{-1} x = T^{-1} x \Rightarrow \lambda^{-1} \in \sigma(T^{-1})$. So,

$\sigma(T)^{-1} \subseteq \sigma(T^{-1})$. It follows that

$$\sigma(T)^{-1} \subseteq \sigma(T^{-1}) = \left(\sigma(T^{-1})^{-1}\right)^{-1} \subseteq \sigma\left((T^{-1})^{-1}\right)^{-1} = \sigma(T)^{-1}$$

$\Rightarrow \sigma(T^{-1}) = \sigma(T)^{-1}$.

Q4

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(x,y,z) = (-x, x-4y+3z, x-6y+5z)$ be a linear transformation.

(a) Find the characteristic polynomial and eigenvalues.

Put $A = M_e^e(T) = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -4 & 3 \\ 1 & -6 & 5 \end{bmatrix}$. Then

$$\Delta(t) = \det(A - tI) = \begin{vmatrix} -1-t & 0 & 0 \\ 1 & -4-t & 3 \\ 1 & -6 & 5-t \end{vmatrix} = -(t+1)\left(-(t+4)(5-t) + 18\right) = -(t+1)(t^2-t-2) = -(t+1)^2(t-2)$$

$$\sigma(T) = \{-1^{(2)}, 2^{(1)}\}$$

(b) Find all eigenspaces (V_λ) and their closures $(\overline{V_\lambda})$.

$\lambda = -1 \Rightarrow$

$$A+1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 3 \\ 1 & -6 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 3 \\ 0 & -3 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = 0,\ y = z \Rightarrow V_{-1} = \operatorname{Span}(0,1,1) = V_{-1,1}$$

$$(A+1)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -9 & 9 \\ 0 & -18 & 18 \end{bmatrix} \Rightarrow y = z$$

$$V_{-1,2} = \{y = z\}$$

Thus $\{0\} \lneqq V_{-1,1} = \{x=0, y=z\} \lneqq V_{-1,2} = \{y=z\} = \overline{V_{-1}}$

Put $f_1 = (1,0,0) \to$ a basis for $V_{-1,2}/V_{-1,1}$

$f_2 = (A+1)f_1 = (0,1,1) \to$ a basis for $V_{-1,1}$

We got the diagram with a single column

f_1

$(A+1)f_1$

$$\lambda = 2 \Rightarrow A - 2 = \begin{bmatrix} -3 & 0 & 0 \\ 1 & -6 & 3 \\ 1 & -6 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 6 & -3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & -3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x = 0 \\ 2y = z \end{array}$$

$$V_2 = \mathrm{Span}\,\underbrace{(0, 1, 2)}_{= f_3}$$

~~Put f_2~~

(c) Find a basis f for $\mathbb{R}^3$ such that $M_f^f(T)$ is in Jordan form.

Put $f = (f_1, f_2, f_3) = ((1,0,0), (0,1,1), (0,1,2))$.

Then $M_f^f(T) = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Q5 Let $T:\mathbb{C}^6\to\mathbb{C}^6$ be a linear transformation such that

$$M_e^e(T)=\begin{bmatrix}0&0&0&0&1&0\\0&0&1&1&0&-1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&1&0&0&-1\\0&0&0&0&0&0\end{bmatrix}$$

with relative to the standard basis e for $\mathbb{C}^6$. Find a basis f for $\mathbb{C}^6$ such that $M_f^f(T)$ is in the Jordan form.

$$\Delta(t)=\begin{vmatrix}-t&0&0&0&1&0\\0&-t&1&1&0&-1\\0&0&-t&0&0&0\\0&0&0&-t&0&0\\0&0&1&0&-t&-1\\0&0&0&0&0&-t\end{vmatrix}=t^2(-t^3)(-t)=t^6$$

$$\sigma(T)=\{0^{⑥}\}$$

$\lambda=0\Rightarrow K_1=\ker(T)=\{x_5=0,\ x_3+x_4=x_6,\ x_4=x_5\}=$

$=\{x_4=x_5=0,\ x_3=x_6\}\Rightarrow \dim(K_1)=3$

Note that $T(x_1,x_2,x_3,x_4,x_5,x_6)=(x_5,x_3+x_4-x_6,0,0,x_3-x_6,0)\Rightarrow$

$\Rightarrow T^2(x_1,x_2,x_3,x_4,x_5,x_6)=(x_3-x_6,0,0,0,0,0)\Rightarrow T^3=0$. So,

$K_2=\ker(T^2)=\{x_3=x_6\}$, $\dim(K_2)=5\Rightarrow K_3=\mathbb{C}^6$. Thus

$\{0\}\underset{\neq}{\subseteq}K_1\underset{\neq}{\subseteq}K_2\underset{\neq}{\subseteq}K_3=\mathbb{C}^6$, $\boxed{\dim(K_3/K_2)=1,\ \dim(K_2/K_1)=2,\ \dim(K_1/\{0\})=3}$

$v_1=(0,0,1,0,0,0)\to$ a basis for K_3/K_2;

$Tv_1=(0,1,0,0,1,0)$, $v_2=(0,0,0,1,0,0)\to$ a basis for K_2/K_1

$T^2v_1=(1,0,0,0,0,0)$, $Tv_2=(0,1,0,0,0,0)$, $v_3=(0,0,1,0,0,1)$

$\to$ a basis for $K_1/\{0\}$.

We got the following diagram

$$\begin{array}{ccc} v_1 & & \\ Tv_1, & v_2 & \\ T^2v_1, & Tv_2, & v_3 \end{array}$$

with three columns. Put

$$f = (\underset{f_1}{\overset{\shortparallel}{v_1}}, \underset{f_2}{\overset{\shortparallel}{Tv_1}}, \underset{f_3}{\overset{\shortparallel}{T^2v_1}}, \underset{f_4}{\overset{\shortparallel}{v_2}}, \underset{f_5}{\overset{\shortparallel}{Tv_2}}, \underset{f_6}{\overset{\shortparallel}{v_3}}).$$

Then

$$M_f^f(T) = \left[\begin{array}{ccc|cc|c} 0 & 0 & 0 & & & \\ 1 & 0 & 0 & & 0 & \\ 0 & 1 & 0 & & & \\ \hline & & & 0 & 0 & \\ & 0 & & 1 & 0 & \\ \hline & & & & & 0 \end{array}\right]$$

Q6

(a) Let $A = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 6 & -3 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & 5 \end{bmatrix}$ be the matrices from M_3. Using the trace inner product from M_3 compute $\langle A, B \rangle$.

$$\langle A, B \rangle = tr(AB^t) = tr\left(\begin{bmatrix} -1 & 2 & 0 \\ 3 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 6 & 0 & 0 \\ -3 & 2 & 0 \\ 1 & -3 & 5 \end{bmatrix}\right) =$$

$$= tr \begin{bmatrix} -12 & 4 & 0 \\ 19 & -3 & 5 \\ 9 & -2 & 0 \end{bmatrix} = -12 - 3 = -15$$

(b) Find the norm (or length) of $2 + x + x^2$ in the inner product space $\mathcal{P}_2(\mathbb{R})$.

$$\|2 + x + x^2\|^2 = \int_{-1}^{1} (2 + x + x^2)^2 dx = \int_{-1}^{1} (4 + 5x^2 + x^4 + 4x + 2x^3)\, dx$$

$$= \int_{-1}^{1} (4 + 5x^2 + x^4)\, dx = 8 + \frac{5}{3} \cdot 2 + \frac{1}{5} \cdot 2$$

(c) Do the vectors $(1, 2, i), (i, 0, 1)$ orthogonal in $\mathbb{C}^3$?

$$\langle (1,2,i), (i,0,1) \rangle = 1 \cdot \bar{i} + 2 \cdot 0 + i \cdot 1 = -i + i = 0.$$

Q7
Using Gram-Schmidt orthogonalization process, find an orthogonal basis for the subspace $V = \{(x,y,z,w) \in \mathbb{R}^4 : x+2y-z-w=0\}$ in $\mathbb{R}^4$.

First put $b_1 = (-2,1,0,0)$, $b_2 = (1,0,1,0)$, $b_3 = (1,0,0,1)$. Then 3

$$a_1 = b_1 = (-2,1,0,0) \quad 1$$

$$a_2 = b_2 - \frac{\langle b_2, a_1\rangle}{\langle a_1, a_1\rangle} a_1 = (1,0,1,0) - \frac{-2}{5}(-2,1,0,0) =$$

$$= \left(\frac{1}{5}, \frac{2}{5}, 1, 0\right) \quad 3$$

$$a_3 = b_3 - \frac{\langle b_3, a_1\rangle}{\langle a_1, a_1\rangle} a_1 - \frac{\langle b_3, a_2\rangle}{\langle a_2, a_2\rangle} a_2 = (1,0,0,1) -$$

$$- \frac{-2}{5}(-2,1,0,0) - \frac{\frac{1}{5}}{\frac{1}{25}+\frac{4}{25}+1}\left(\frac{1}{5}, \frac{2}{5}, 1, 0\right)$$

5

$$= (1,0,0,1) + \left(-\frac{4}{5}, \frac{2}{5}, 0, 0\right) - \frac{1}{6}\left(\frac{1}{5}, \frac{2}{5}, 1, 0\right)$$

$$= \left(\frac{1}{5}, \frac{2}{5}, 0, 1\right) - \left(\frac{1}{30}, +\frac{1}{15}, +\frac{1}{6}, 0\right)$$

$$= \left(\frac{1}{6}, \frac{1}{3}, \frac{-1}{6}, 1\right)$$

Thus (a_1, a_2, a_3) is an orthogonal basis for V.

Q8) Let V be a finite dimensional vector space such that $V = \mathcal{U} \oplus W$ for subspaces $\mathcal{U}, W \leq V$. Thus $\mathcal{U} \cap W = \{0\}$ and $V = \mathcal{U} + W$. Say e is a basis for $\mathcal{U}$ and f is a basis for W. Prove that $e \cup f$ is a basis for V.

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Put $e = (e_1, \dots e_r)$, $f = (f_1, \dots f_s)$ with $r = \dim(\mathcal{U})$, $s = \dim(W)$. Since $V = \mathcal{U} + W$, it follows that $V = \text{Span}(e) + \text{Span}(f) =$
$= \text{Span}(e \cup f)$.

Assume $\sum_{i=1}^{r} \lambda_i e_i + \sum_{j=1}^{s} \mu_j f_j = 0$. Then $\sum \lambda_i e_i = \sum (-\mu_j) f_j \in V \cap W = \{0\}$, which in turn implies that $\lambda_i = 0$, $\mu_j = 0$ for all i, j.

Thus $e \cup f$ are lin. independent vectors.