	Basic Linear Algebra II. Midterm	
Code : *Math 260* Acad.Year: *2010-2011* Semester : *Spring* Date : *7.5.2011* Time : *10:00* Duration : *120 minutes*	Last Name: Name : Student No: Department: Section: Signature:	
	6 QUESTIONS ON 6 PAGES TOTAL 100 POINTS	

1. (14)	2. (16)	3. (16)	4. (24)	5. (21)	6. (9)		

Show your work! No calculators! Please draw a $\boxed{\text{box}}$ around your answers!
Please do not write on your desk!

1. (14 pts) The following parts are about linear transformations.

(a) (4 pts) Let T be $90°$ rotation around the x-axis in $\mathbb{R}^3$ (taking the positive y-axis to the positive z-axis). Write the matrix $A = [T]$ of the transformation T.

$$T\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad T\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\-1\\0\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$A=[T]=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}$$

(b) (4 pts) Write the matrix B which reflects $\mathbb{R}^3$ across the xy-plane.

$$B=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$$

$$BA=\begin{bmatrix}1&0&0\\0&0&-1\\0&-1&0\end{bmatrix}$$

(c) (2 pts) Compute $T_{BA}(x, y, z)$ (T_{BA} is the the transformation whose matrix is BA).

$$T_{BA}(x,y,z) = \left(\boxed{x}, \boxed{-z}, \boxed{-y}\right)$$

(d) (4 pts) Geometrically describe the action of the linear transformation T_{BA}.
(i.e. Is it a reflection (across what?), or rotation (how much?), or scaling, etc.?)

Reflection across the plane $y+z=0$

2. (16 pts) The following parts all deal with the vectors $\mathcal{B} = \{v_1, v_2, v_3\}$ where $v_1 = (1, -1, 0)$, $v_2 = (1, 0, 2)$, and $v_3 = (0, 0, 1)$.

(a) (4 pts) Show that $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

rank = 3 $\Rightarrow$ independent.

(b) (4 pts) Find the coordinates of $(4, 2, 5)$ relative to the basis $\mathcal{B}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ -1 & 0 & 0 & 2 \\ 0 & 2 & 1 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 6 \\ 0 & 2 & 1 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -7 \end{array}\right]$$

So, coordinates of $(4,2,5)$ are $\begin{bmatrix} -2 \\ 6 \\ -7 \end{bmatrix}$

(c) (4 pts) Consider the inner product on $\mathbb{R}^3$ coming from $\mathcal{B}$ by setting $\langle v_i, v_j \rangle = 0$ for all $i \neq j$ and $\langle v_i, v_i \rangle = 1$ for each i.
What is the norm of $(4, 2, 5)$ with respect to this inner product?

$(4,2,5) = -2v_1 + 6v_2 - 7v_3$ $\quad \langle v_1, v_1 \rangle = 1$, $\langle v_1, v_2 \rangle = 0$ etc.

$\langle (4,2,5), (4,2,5) \rangle = (-2)^2 + 6^2 + (-7)^2 = 89$

$\|(4,2,5)\| = \sqrt{89}$

(d) (4 pts) Compute $\langle (4, 2, 5), (1, 0, 0) \rangle$ using the inner product from (c).

$(1,0,0) = v_2 - 2v_3$

$\langle (4,2,5), (1,0,0) \rangle = \langle -2v_1 + 6v_2 - 7v_3, v_2 - 2v_3 \rangle$

$= 6 + 14$

$= 20$

Note: $\langle\ ,\ \rangle$ above is the inner product associated to the matrix

$$A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \downarrow & \downarrow & \downarrow \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}^{-1}$$

because $A^{-1}\bar{v}_i = \bar{e}_i$ so $(A^{-1}\bar{v}_i) \cdot (A^{-1}\bar{v}_j) = \bar{e}_i \cdot \bar{e}_j = 0$

$(A^{-1}\bar{v}_i) \cdot (A^{-1}\bar{v}_i) = \bar{e}_i \cdot \bar{e}_i = 1$

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3. (16 pts) For each part below, decide which of the following is true:

I. V is not a vector space.
II. V is a vector space but W is not a subspace.
III. V is a vector space and W is a subspace.

If V is not a vector space, then indicate why not. Either show W is a subspace, or give an example why it isn't.

(a) **(4 pts)** $V = \mathbb{R}^3$ with the usual operations. $W = \{(a,b,c) \mid \ a - 2b + c = 0\}$.

V is a vector space

$(a,b,c), (a',b',c') \in W \Rightarrow \begin{matrix} a-2b+c=0 \\ a'-2b'+c'=0 \end{matrix} \Rightarrow (a+a') - 2(b+b') + (c+c') = 0$

$\Rightarrow (a,b,c) + (a',b',c') \in W$

$(a,b,c) \in W, k \in \mathbb{R} \Rightarrow a - 2b + c = 0 \Rightarrow ka - 2kb + kc = 0 \Rightarrow k(a,b,c) \in W$

$W \neq \emptyset$ since $(0,0,0) \in W$. So W is a subspace

III

(b) **(4 pts)** $V = \mathbb{Z}$ with the usual operations. W is the set of all even integers.

V is not a vector space: $1 \in V$ but $\frac{1}{2} \cdot 1 \notin V$.

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(c) **(4 pts)** V is continuous functions on $\mathbb{R}$ with the usual operations. W is the subset of positive functions (i.e. $f(x) \geq 0$).

V is a vector space.

W is not a subspace: $x^2 \in W$. But $(-1)x^2 \notin W$.

II

(d) **(4 pts)** $V = \mathbb{R}^2_+$ (i.e. $V = \{(x,y) \mid \ x > 0,\, y > 0\}$) with the operations $(x,y) \oplus (z,w) = (xz, yw)$ and $k \otimes (x,y) = (x^k, y^k)$. W is the set of points (x,y) on the parabola $y = x^2$ in V.

V is a vector space.

$(x,y), (z,w) \in W \Rightarrow \begin{matrix} y = x^2 \\ w = z^2 \end{matrix} \Rightarrow yw = (xz)^2 \Rightarrow (x,y) \oplus (z,w) \in W$

$(x,y) \in W, k \in \mathbb{R} \Rightarrow y = x^2 \Rightarrow y^k = (x^2)^k = (x^k)^2 \Rightarrow k \otimes (x,y) \in W$

$W \neq \emptyset$ since $(1,1) \in W$.

So W is a subspace.

III

4. **(24 pts)** Give a short (one sentence) answer for each of the following parts.

(a) None of the following sets of vectors span $\mathcal{P}_2$ (the vector space of degree 2 polynomials). For each, give a simple reason why not.

$\dim \mathbb{P}_2 = 3$

(i) (3 pts) $\{1+x^2,\ 1+x\}$

number of vectors $= 2 < \dim P_2 = 3$.

(ii) (3 pts) $\{0,\ 1+x,\ 1+x+x^2\}$

0 does not change the span, and $\dim(\text{span}(1+x, 1+x+x^2)) = 2 < 3$

(iii) (3 pts) $\{1+x,\ x+x^2,\ 2+2x,\ 3x+3x^2\}$

$2+2x = 2(1+x)$ and $3x+3x^2 = 3(x+x^2)$, so

$\text{Span}(1+x, x+x^2, 2+2x, 3x+3x^2) = \text{Span}(1+x, x+x^2) \Rightarrow$ dim of the span $= 2 < 3$.

(iv) (3 pts) $\{x,\ x+x^2,\ 2x+3x^2,\ -x+2x^2\}$

No polynomial has a constant term, so 1 cannot be a linear combination of them.

(b) None of the following sets of symmetric 2×2 matrices are linearly independent. For each, give a simple reason why not.

Sym. 2×2 matrices: $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

A basis: $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

$\dim = 3$

(i) (3 pts) $\left\{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\right\}$

4 vectors in dim 3

$\Rightarrow$ linearly dependent.

(ii) (3 pts) $\left\{\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 3 & 0 \end{bmatrix}\right\}$

3 vectors lie in the proper subspace $c=0$ ($\Rightarrow \dim = 2$), so linearly dependent.

(iii) (3 pts) $\left\{\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix}\right\}$

$\begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix} = 2\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

(iv) (3 pts) $\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$

Any set containing $\vec{0}$ is linearly dependent.

5. **(21 pts)** The following parts deal with the matrix A which reduces as given below:

$$A = \begin{bmatrix} 2 & 3 & -1 & 1 & -1 & 1 & 2 \\ 1 & 0 & 1 & 2 & 4 & 2 & 4 \\ 3 & 3 & 0 & 3 & 3 & 3 & 6 \\ 0 & -1 & 1 & 1 & 3 & 3 & 8 \\ -1 & 3 & -4 & 3 & -5 & 4 & 1 \end{bmatrix}$$

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & -1 & 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \cdot 2s - 2t - 2$$

(a) (5 pts) Give a basis for the column space of A.

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 4 \end{bmatrix} \right\}$$

(b) (2 pts) Write a nontrivial (some constants nonzero) relation among the column vectors of A:

$0 = \boxed{-1}(\text{Col1}) + \boxed{1}(\text{Col2}) + \boxed{1}(\text{Col3}) + \boxed{0}(\text{Col4}) + \boxed{0}(\text{Col5}) + \boxed{0}(\text{Col6}) + \boxed{0}(\text{Col7})$

(c) (5 pts) Give a basis for the row space of A.

$$\{ [1\ 0\ 1\ 0\ 2\ 0\ 2], [0\ 1\ -1\ 0\ -2\ 0\ -1], [0\ 0\ 0\ 1\ 1\ 0\ -2], [0\ 0\ 0\ 0\ 0\ 1\ 3] \}$$

(d) (5 pts) Give a basis for the null space of A.

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 2 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$

(e) (2 pts) What is the rank of A?

$\operatorname{rank} A = 4$

(f) (2 pts) Do the columns of A span $\mathbb{R}^5$? Explain.

No, since $\dim(\text{col. space}) = \operatorname{rank} A = 4 < \dim(\mathbb{R}^5) = 5$

6. (9 pts) Consider an electrical circuit formed by a 2×2 grid. The standard way to write the KVL equations for this circuit is to write them for the four small loops around grid squares, as in figure 1. Do we get the same result if we replace these equations by the following four: the two 1×2 loops and the two 2×1 loops, as in figure 2? Explain.

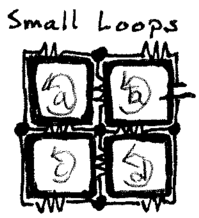

Figure 1.

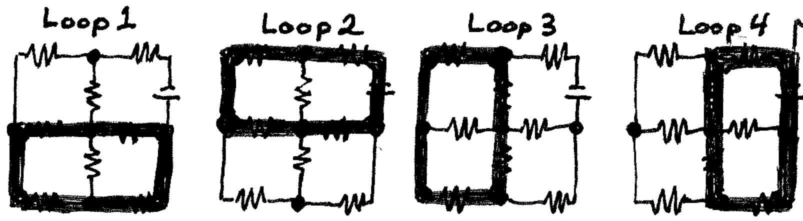

Figure 2.

Say the first four loops give

$$a = 0$$
$$b = 0$$
$$c = 0$$
$$d = 0$$

Those 4 equations are independent.

The second 4 are

$$c + d = 0$$
$$a + b = 0$$
$$a + c = 0$$
$$b + d = 0.$$

Also note:
- (loop 1) + (loop 2) = ⊞
- (loop 3) + (loop 4) = ⊞

so (loop 1) + (loop 2) − (loop 3) − (loop 4) = 0

$\Rightarrow$ Not linearly independent!!

The second set can clearly be obtained from the first. But;

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix has rank $3 \Rightarrow$ not invertible $\Rightarrow$ the first set cannot be obtained from the second. So we do not get the same result.