M E T U
Northern Cyprus Campus

	Basic Linear Algebra I. Midterm	
Code : *Math 260* Acad.Year: *2010-2011* Semester : *Spring* Date : *2.4.2011* Time : *10:00* Duration : *120 minutes*	Last Name: Name : Department: Signature: 7 QUESTIONS ON 7 PAGES TOTAL 100 POINTS	Student No: Section:

1	2	3	4	5	6	7		

Show your work! No calculators! Please draw a $\boxed{\text{box}}$ around your answers!
Please do not write on your desk!

1. (8+4+6) This problem has three parts. In all of these parts the matrix A is given by

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 0 \\ 4 & -3 & 8 \end{bmatrix}.$$

(a) Find 3×3 matrices L and U so that L is lower triangular, U is upper triangular with diagonal entries 1, and $LU = A$.

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 0 \\ 4 & -3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 0 \\ 4 & -3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 0 & 3 & -6 \\ 4 & -3 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 4 & -3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 5 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

(b) What is the determinant of A?

$$\det(A) = \det(LU) = \det(L) \cdot \det(U) = 18$$

(c) Use the matrices L and U to solve $A\mathbf{x} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$

$LUx = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$. Say $Ux = y$. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$

$\Rightarrow y_1 = -2$, $2y_1 + 3y_2 = -1 \Rightarrow y_2 = 1$, $4y_1 + 5y_2 + 6y_3 = 3 \Rightarrow y_3 = 1$

$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \boxed{x_3 = 1}$, $x_2 - 2x_3 = 1 \Rightarrow \boxed{x_2 = 3}$, $x_1 - 2x_2 + 3x_3 = -2$

$\Rightarrow \boxed{x_1 = 1}$

2. (4+4+4+4) The following parts involve the steps needed to solve a system of equations. (Note: each part involves a different system!)

(a) Convert the following system of equations to an augmented matrix

$$x + z = 3$$
$$y - z = 0$$
$$x + y = 1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

(b) Reduce the following augmented matrix to row echelon form (but do not solve!).

$$\left[\begin{array}{ccc|c} 0 & 1 & 3 & 1 \\ 2 & 4 & -2 & 4 \\ 3 & 6 & 2 & -1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 4 & -2 & 4 \\ 0 & 1 & 3 & 1 \\ 3 & 6 & 2 & -1 \end{array}\right] \xrightarrow{\frac{R_1}{2} \to R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 3 & 1 \\ 3 & 6 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{-3R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 5 & -7 \end{array}\right] \xrightarrow{\frac{R_3}{5} \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & -7/5 \end{array}\right]$$

(c) Reduce the following matrix to redued row echelon form (but do not solve!).

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_2 + R_1 \to R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{-3R_3 + R_1 \to R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-R_3 + R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

(d) Give the solution of the augmented matrix

$$\left[\begin{array}{cccccc|c} 1 & 2 & -1 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 + R_1 \to R_1} \left[\begin{array}{cccccc|c} 1 & 2 & 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & -1 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 + R_2 \to R_2]{R_3 + R_1 \to R_1} \begin{array}{c} \left[\begin{array}{cccccc|c} 1 & 2 & 0 & 1 & 0 & 4 & -2 \\ 0 & 0 & 1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccccc} & \uparrow & & \uparrow & & \uparrow \\ & \text{free} & & \text{free} & & \text{free} \end{array} \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2 - 2s - t - 4u \\ s \\ -1 + t - 3u \\ t \\ -3 - u \\ u \end{bmatrix} \qquad t, u, s \in \mathbb{R}$$

3. (8) Find the inverse (if it exists) of the following matrix using row reduction (Gauss-Jordan elimination):

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{R_1}{2} \to R_1} \left[\begin{array}{ccc|ccc} 1 & -1/2 & 0 & 1/2 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 + R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & -1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{2}{3} R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & -1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & -2/3 & 1/3 & 2/3 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2/2 + R_1 \to R_1]{R_2 + R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1/3 & 2/3 & 1/3 & 0 \\ 0 & 1 & -2/3 & 1/3 & 2/3 & 0 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{3}{4} R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1/3 & 2/3 & 1/3 & 0 \\ 0 & 1 & -2/3 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right]$$

$$\xrightarrow[2R_3/3 + R_2 \to R_2]{R_3/3 + R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right]$$

$$\underbrace{\begin{array}{ccc} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{array}}_{A^{-1}}$$

4. (4+4+4) Consider the following system

$$x^3 + y^2 = 1$$
$$3x^3 + ky^2 = \ell$$

For which k, ℓ does this system have:

(a) **Finitely many** real solutions?

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & k & \ell \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & k-3 & \ell-3 \end{array}\right]$$

If $k \neq 3 \Rightarrow y^2 = \frac{\ell-3}{k-3}$. This needs to be positive.

If $\boxed{k > 3 \text{ and } \ell > 3} \Rightarrow$ 2 solns. for y, and $x^3 = 1 - y^2$
$\Rightarrow$ 1 soln. for x

If $\boxed{k < 3 \text{ and } \ell < 3} \Rightarrow$ similar

(b) **No** real solutions?

If $\boxed{k > 3 \text{ and } \ell < 3}$ or $\boxed{k < 3 \text{ and } \ell > 3}$, then, no solutions.

Also, if $\boxed{k = 3 \text{ and } \ell \neq 3}$, then no solutions exist.

(c) **Infinitely many** real solutions?

Need $\boxed{k = 3 \text{ and } \ell = 3}$.

In this case y^2 can take any positive value, and $x^3 = 1 - y^2$.

5. (8+4+4+4) The following parts deal with determinants.

(a) Write a formula involving the determinant of a matrix which gives the equation of the parabola $ay + bx^2 + cx + d = 0$ through the points $(1, -1)$, $(2, 4)$, $(-1, 0)$. (Do not compute the determinant - just write the formula)

$$\begin{vmatrix} y & x^2 & x & 1 \\ -1 & 1 & 1 & 1 \\ 4 & 4 & 2 & 1 \\ 0 & 1 & -1 & 1 \end{vmatrix} = 0$$

(b) Suppose A and B are 10×10 matrices with $\det(A) = 3$, $\det(B) = 5$. What is $\det\left((2A^2)^{-1}B^{\mathrm{T}}\right)^{-1}$?

$$\det\left((2A^2)^{-1}B^T\right)^{-1} = \frac{1}{\det\left((2A^2)^{-1}\right)\det B^T}$$

$$= \frac{\det\left((2A^2)^{+1}\right)}{\det B}$$

$$= \frac{2^{10}(\det A)^2}{\det B} = \boxed{\frac{2^{10} \cdot 3^2}{5}}$$

(c) Write the 3×3 elementary matrix which performs the row operation "Replace row 3 by row 3 added to -5 times row 1".

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$$

(d) Write the determinant of the matrix from (c).

$$\boxed{\det E = 1}$$

6. (8+8) The following parts deal with the matrix $A = \begin{bmatrix} 0 & 1 & 14 & 0 & -1 & 7 & -3 \\ 3 & 1 & -1 & 3 & 7 & 12 & 15 \\ 0 & 0 & 2 & -1 & -8 & -4 & 1 \\ 0 & 0 & 0 & -1 & 12 & 7 & -1 \\ 0 & 0 & 0 & 0 & -2 & 5 & 12 \\ 0 & 0 & 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}$

(a) What is the number is in row 4, column 4 of A^{-1}?

$$(A^{-1})_{44} = \frac{C_{44}}{\det A} = \frac{M_{44}}{\det A}$$

$$= \frac{\begin{vmatrix} 0 & 1 & & & & \\ 3 & 1 & & & * & \\ & & 2 & & & \\ & & & -2 & & \\ & 0 & & & 3 & \\ & & & & & 7 \end{vmatrix}}{\begin{vmatrix} 0 & 1 & & & & & \\ 3 & 1 & & & * & & \\ & & 2 & & & & \\ & & & -1 & & & \\ & & & & -2 & & \\ & 0 & & & & 3 & \\ & & & & & & 7 \end{vmatrix}} = \frac{-\begin{vmatrix} 3 & 1 & & & & \\ 0 & 1 & & & * & \\ & & 2 & & & \\ & & & -2 & & \\ & 0 & & & 3 & \\ & & & & & 7 \end{vmatrix}}{-\begin{vmatrix} 3 & 1 & & & & & \\ 0 & 1 & & & * & & \\ & & 2 & & & & \\ & & & -1 & & & \\ & & & & -2 & & \\ & 0 & & & & 3 & \\ & & & & & & 7 \end{vmatrix}}$$

triangular

$$= \frac{-3\cdot 1\cdot 2(-2)\cdot 3\cdot 7}{-3\cdot 1\cdot 2\cdot(-1)(-2)\cdot 3\cdot 7}$$

$$\boxed{= -1}$$

(b) Use Cramer's Rule to find the x_5 part of the solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 0 \\ 0 \end{bmatrix}$

$$X_5 = \frac{\det(A_5)}{\det A}$$

$$= \frac{\begin{vmatrix} 0 & 1 & & & & & \\ 3 & 1 & & & * & & \\ & & 2 & & & & \\ & & & -1 & & & \\ & & & & 5 & & \\ & 0 & & & & 3 & \\ & & & & & & 7 \end{vmatrix}}{\begin{vmatrix} 0 & 1 & & & & & \\ 3 & 1 & & & * & & \\ & & 2 & & & & \\ & & & -1 & & & \\ & & & & -2 & & \\ & 0 & & & & 3 & \\ & & & & & & 7 \end{vmatrix}}$$

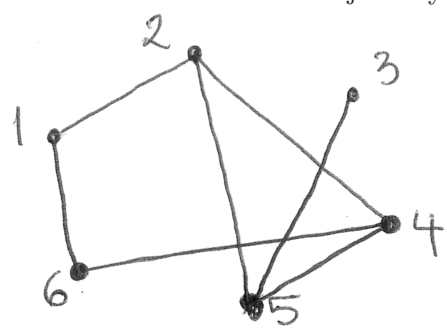
(as above)

$$= \frac{-3\cdot 1\cdot 2\cdot(-1)\cdot 5\cdot 3\cdot 7}{-3\cdot 1\cdot 2\cdot(-1)(-2)\cdot 3\cdot 7}$$

$$\boxed{= -5/2}$$

7. (5+5) The following parts deal with the vertex adjacency matrix $M = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

(a) Draw the graph which has this vertex adjacency matrix.



(b) By looking at the graph, figure out the number which is in row 3 and column 1 of M^4.

Need to count the number of paths of length 4 (4 edges) from 3 to 1.

Two such paths:

$3 \longrightarrow 5 \longrightarrow 4 \longrightarrow 6 \longrightarrow 1$

$3 \longrightarrow 5 \longrightarrow 4 \longrightarrow 2 \longrightarrow 1$

So $\boxed{(M^4)_{31} = 2}$