M E T U
Northern Cyprus Campus

<table>
<tr><td colspan="2">Basic Linear Algebra
II. Midterm</td></tr>
<tr><td>Code : Math 260
Acad.Year: 2009-2010
Semester : Spring
Date : 2.5.2010
Time : 9:00
Duration : 120 minutes</td><td>Last Name:
Name :   Student No
Department: Section:
Signature:

6 QUESTIONS ON 6 PAGES
TOTAL 100 POINTS</td></tr>
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1	2	3	4	5	6		

1. (5+5+5=15 points) (a) Find all values of k such that the vectors $(1,1,k)$ and $(-3k,2,k)$ are orthogonal.

$(1,1,k)\cdot(-3k,2,k)=0$

$-3k+2+k^2=0$

$(k-2)(k-1)=0$

$k=1$ or $k=2$

(b) Show that if $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal vectors, then the norms of $\mathbf{u}$ and $\mathbf{v}$ must be equal.

$(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v})=0$

$\Rightarrow \vec{u}\cdot\vec{u}+\cancel{\vec{v}\cdot\vec{u}}-\cancel{\vec{u}\cdot\vec{v}}-\vec{v}\cdot\vec{v}=0$

$\Rightarrow \vec{u}\cdot\vec{u}=\vec{v}\cdot\vec{v}$

$\Rightarrow \|\vec{u}\|^2=\|\vec{v}\|^2$

$\Rightarrow \|\vec{u}\|=\|\vec{v}\|$

(c) Prove the inequality $x_1+x_2+\ldots+x_n \leq \sqrt{n}\sqrt{x_1^2+\ldots+x_n^2}$ by applying the Cauchy-Schwarz inequality to two suitable vectors in $\mathbb{R}^n$.

Take $\vec{u}=(1,1,\ldots,1)$ and $\vec{v}=(x_1,\ldots,x_n)$.

Since $\vec{u}\cdot\vec{v}\leq\|\vec{u}\|\,\|\vec{v}\|$,

$(1,1,\ldots,1)\cdot(x_1,\ldots,x_n)\leq\sqrt{1^2+\cdots+1^2}\sqrt{x_1^2+\cdots+x_n^2}$

$x_1+x_2+\cdots+x_n\leq\sqrt{n}\sqrt{x_1^2+\cdots+x_n^2}$

2. (20 points) Find the eigenvalues and eigenvectors of the linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ determined by

$$\begin{aligned} u &= 5x + 2z \\ v &= 3y \\ w &= 2x + 5z \end{aligned}$$

Matrix: $A = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 5 \end{bmatrix}$

$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 5-\lambda & 0 & 2 \\ 0 & 3-\lambda & 0 \\ 2 & 0 & 5-\lambda \end{vmatrix} = 0$

$\iff (3-\lambda) \begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = 0$

$\iff (3-\lambda)\left[(5-\lambda)^2 - 4\right] = 0 \iff (3-\lambda)(3-\lambda)(7-\lambda) = 0$

$\iff \boxed{\lambda = 3}$ or $\boxed{\lambda = 7}$

<u>eigenvectors:</u>

<u>$\lambda = 3$:</u>

$(A - \lambda I)\vec{v} = \vec{0}$

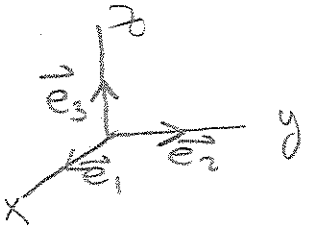
$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -t \\ s \\ t \end{bmatrix}$ $t, s \in \mathbb{R}$, not both 0.

<u>$\lambda = 7$</u>

$(A - \lambda I)\vec{v} = \vec{0}$

$\left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 0 & -4 & 0 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}$ $t \in \mathbb{R}$, $t \neq 0$.

3. (15 points) Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ obtained by rotation about the z-axis by $30°$ (from the $+x$-axis towards the $+y$-axis), followed by a reflection across the plane $y = 0$, followed by a rotation about the y-axis by $90°$ (from the $+z$-axis towards the $+x$-axis). Find the standard matrix for the **inverse** of T. (Hint: Find out what happens to the standard basis vectors $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ under T^{-1}.)



T_1: rotation about z-axis by $30°$
T_2: reflection across $y = 0$
T_3: rotation about y-axis by $90°$

$$T = T_3 \circ T_2 \circ T_1$$
$$\Rightarrow T^{-1} = T_1^{-1} \circ T_2^{-1} \circ T_3^{-1}$$

$$T^{-1}(\vec{e}_1) = T_1^{-1}(T_2^{-1}(T_3^{-1}(\vec{e}_1))) = T_1^{-1}(T_2^{-1}(\vec{e}_3))$$
$$= T_1^{-1}(\vec{e}_3)$$
$$= \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T^{-1}(\vec{e}_2) = T_1^{-1}(T_2^{-1}(T_3^{-1}(\vec{e}_2))) = T_1^{-1}(T_2^{-1}(\vec{e}_2))$$
$$= T_1^{-1}(-\vec{e}_2)$$
$$= \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \\ 0 \end{bmatrix}$$

$$T^{-1}(\vec{e}_3) = T_1^{-1}(T_2^{-1}(T_3^{-1}(\vec{e}_3))) = T_1^{-1}(T_2^{-1}(-\vec{e}_1))$$
$$= T_1^{-1}(-\vec{e}_1)$$
$$= \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \text{standard matrix for } T^{-1} = \left[T^{-1}(\vec{e}_1) \,\middle|\, T^{-1}(\vec{e}_2) \,\middle|\, T^{-1}(\vec{e}_3) \right] = \begin{bmatrix} 0 & -1/2 & -\sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}$$

4. (10+10=20 points) Find the quadratic interpolant $y = ax^2 + bx + c$ passing through the points $(1, 2), (2, -3)$ and $(3, 10)$ by applying

(a) standart interpolation (Vandermonde matrix),

$$y = ax^2 + bx + c$$

$$\begin{aligned} a + b + c &= 2 \\ 4a + 2b + c &= -3 \\ 9a + 3b + c &= 10 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & -3 \\ 9 & 3 & 1 & 10 \end{array}\right] \xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 3 & 1 & 0 & -5 \\ 8 & 2 & 0 & 8 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 3 & 1 & 0 & -5 \\ 2 & 0 & 0 & 18 \end{array}\right]$$

$\Rightarrow a = 9$

$27 + b = -5$

$\Rightarrow b = -32$

$a + b + c = 2$

$\Rightarrow c = 25$

So,

$y = 9x^2 - 32x + 25$

(b) Newton's interpolation.

$$y = a(x-1)(x-2) + b(x-1) + c$$

$$\left.\begin{aligned} 2 &= y(1) = c \\ -3 &= y(2) = b + c \\ 10 &= y(3) = 2a + 2b + c \end{aligned}\right\} \quad c = 2, \ b = -5, \ a = 9$$

So, $y = 9(x-1)(x-2) - 5(x-1) + 2$

$= 9x^2 - 32x + 25$

5. (18 points) Let V be the set of all pairs (x, y) of real numbers with $x > 0$ and $y > 0$. Suppose we define two operations $+$ and $\cdot$ by $(x, y) + (x', y') = (xx', yy')$ and $k \cdot (x, y) = (x^k, y^k)$. Below are listed the 10 vector space axioms. For each axiom, state whether it is satisfied or not, by giving your reasoning. Finally, also indicate whether V is a vector space or not under these operations.

(1) If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V. ✓

$(x, y) + (x', y') = (xx', yy')$

If $x, x', y, y' > 0 \Rightarrow xx' > 0,\ yy' > 0 \Rightarrow$ sum is in V.

(2) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

$(x,y) + (x',y') = (x',y') + (x,y) = (xx', yy')$ ✓

(3) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

$(x,y) + ((x',y') + (x'',y'')) = (xx'x'', yy'y'') = ((x,y) + (x',y')) + (x'',y'')$ ✓

(4) There is an object $\mathbf{0}$ in V called a *zero vector* for V, such that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$ in V. Need $(a,b) + (x,y) = (x,y)$ for all x, y.

$\Leftrightarrow$ $ax = x$, $by = y$ for all x, y. Take $0 = (1,1) \in V$ ✓

(5) For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a *negative* of $\mathbf{u}$, such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.

Need $(x,y) + (x',y') = (1,1)$

$xx' = 1,\ yy' = 1 \Rightarrow x' = \frac{1}{x},\ y' = \frac{1}{y}$ ✓ $\left(\frac{1}{x}, \frac{1}{y}\right) \in V$

(6) If k is any scalar and $\mathbf{u}$ is any object in V, then $k \cdot \mathbf{u}$ is in V.

$k \cdot (x,y) = (x^k, y^k)$. If $x, y > 0 \Rightarrow x^k, y^k > 0$. ✓

(7) $k \cdot (\mathbf{u} + \mathbf{v}) = k \cdot \mathbf{u} + k \cdot \mathbf{v}$.

$k \cdot ((x,y) + (x',y')) = k \cdot (xx', yy') = ((xx')^k, (yy')^k)$

$= (x^k x'^k, y^k y'^k) = (x^k, y^k) + (x'^k, y'^k)$

$= k \cdot (x,y) + k \cdot (x',y')$

(8) $(k + m) \cdot \mathbf{u} = k \cdot \mathbf{u} + m \cdot \mathbf{v}$.

$(k+m) \cdot (x,y) = (x^{k+m}, y^{k+m})$

$= (x^k x^m, y^k y^m) = (x^k, y^k) + (x^m, y^m)$ ✓

$= k \cdot (x,y) + m \cdot (x,y)$.

(9) $k \cdot (m \cdot \mathbf{u}) = (km) \cdot \mathbf{u}$.

$k \cdot (x^m, y^m) = (x^{mk}, y^{mk})$ ✓

(10) $1 \cdot \mathbf{u} = \mathbf{u}$.

$1 \cdot (x,y) = (x^1, y^1)$ ✓

(Conclusion) Choose one: V is / is not a vector space.

6. (12 points) Determine whether the given W is a subspace of the vector space V or not. (You do not have to check that V is a vector space. The operations on V below are the usual operations in each case. The different parts of this question are not related to one another)

(a) V is the set of all 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and W is the subset of all matrices with $a + d = 0$ (trace 0 matrices).

$W \neq \phi$ since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$.

Say $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in W$.

Then $\left.\begin{matrix} a+d=0 \\ a'+d'=0 \end{matrix}\right\} a+a'+d+d'=0 \Rightarrow \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} \in W$ ✓

Say $k \in \mathbb{R}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$.

$ka + kd = k(a+d) = 0$

$\Rightarrow k \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$

So W is a subspace of V.

(b) V is the set of all 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and W is the subset of all matrices with determinant equal to 1.

W is not a subspace: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in W$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in W$

But $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \notin W$.

(c) $V = \mathbb{R}^3$, and W is the set of all (x, y, z) with $x + 2y - z = 2$.

W is not a subspace:

$(x, y, z) = (0, 1, 0) \in W$.

$(x, y, z) = (2, 0, 0) \in W$

But $(2, 0, 0) + (0, 1, 0) = (2, 1, 0) \notin W$

(d) V is the set of all polynomials $p(x)$ of degree at most 3, and W is the subset of all $p(x)$ such that $p(2) = 0$.

- $W \neq \phi$; take $p(x) = x - 2$.
- Say $p, q \in W$. $(p+q)(x) = p(x) + q(x)$

  So if $p(2) = q(2) = 0 \Rightarrow (p+q)(2) = 0 \Rightarrow p+q \in W$ ✓
- Say $p \in W$, $k \in \mathbb{R}$. $(kp)(x) = k \cdot p(x)$

  So if $p(2) = 0 \Rightarrow (kp)(2) = 0 \Rightarrow kp \in W$ ✓