M E T U
Northern Cyprus Campus

Basic Linear Algebra
I. Midterm

Code : *Math 260*
Acad.Year: *2009-2010*
Semester : *Spring*
Date : *4.4.2010*
Time : *9:00*
Duration : *120 minutes*

Last Name:
Name :
Student No
Department:
Section:
Signature:

5 QUESTIONS ON 6 PAGES
TOTAL 100 POINTS

1	2	3	4	5		

1. (15+10=25 points) Consider the linear system of equations

$$\begin{aligned} y + az &= 1 \\ x + y + z &= b \\ x - z &= a \end{aligned}$$

(a) Find the values of a and b for which there is

- a unique solution,
- no solutions,
- infinitely many solutions.

(You do not actually have to find the solutions.)

$$\left[\begin{array}{ccc|c} 0 & 1 & a & 1 \\ 1 & 1 & 1 & b \\ 1 & 0 & -1 & a \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 1 & a & 1 \\ 1 & 0 & -1 & a \end{array}\right] \xrightarrow{-R_1+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 1 & a & 1 \\ 0 & -1 & -2 & a-b \end{array}\right]$$

$$\xrightarrow{R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 1 & a & 1 \\ 0 & 0 & a-2 & a-b+1 \end{array}\right]$$

① $a=2$, $a-b+1 \neq 0$ no solution ($b \neq 3$)
② $a=2$, $b=3$ infinitely many solutions
③ $a \neq 2$ unique solution

(b) For the values of $a = b = 0$, find the solution of the system using Cramer's rule.

$$\det A = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = -1(-1-1) = 2$$

$$x = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{vmatrix}}{2} = \frac{-1}{2}, \quad y = \frac{\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{vmatrix}}{2} = \frac{-1(-1-1)}{2} = 1$$

$$z = \frac{\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}}{2} = \frac{-1}{2} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

2. (5+5+5+5=20 points) Evaluate the following determinants. Show your work. Please use the method specified in brackets, if there is one.

(a) $\begin{vmatrix} 3 & -1 & 0 \\ -2 & 1 & 4 \\ 6 & 6 & 3 \end{vmatrix}$ (use cofactor expansion).

$$3\begin{vmatrix} 1 & 4 \\ 6 & 3 \end{vmatrix} - (-1)\begin{vmatrix} -2 & 4 \\ 6 & 3 \end{vmatrix} = 3(-21) - (-1)(-30)$$

$$= -93$$

(b) $\begin{vmatrix} 1 & 3 & -1 & 1 \\ -1 & -2 & 6 & 2 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 1 & -5 \end{vmatrix}$ (use row reduction).

$$\overset{R_1+R_2\to R_2}{=} \begin{vmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 5 & 3 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 1 & -5 \end{vmatrix} \overset{-3R_2+R_3\to R_3}{=} \begin{vmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & -10 & -9 \\ 0 & 0 & 1 & -5 \end{vmatrix}$$

$$\overset{R_3\leftrightarrow R_4}{=} -\begin{vmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & -10 & -9 \end{vmatrix} \overset{10R_3+R_4\to R_4}{=} -\begin{vmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & -59 \end{vmatrix}$$

$$= 59$$

(c) $\begin{vmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{vmatrix}$. $\overset{\substack{R_1 \longleftrightarrow R_4 \\ R_2 \longleftrightarrow R_5 \\ R_3 \longleftrightarrow R_6}}{=\!=}$ $(-1)^3 \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$

$= \underline{-1}$

(d) $\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 & 2 & 4 & 3 \\ 4 & -2 & 5 & 7 & 8 & 9 & 4 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 5 & 4 & 5 & 7 & 7 & 2 & 2 \\ -1 & 5 & -2 & 0 & 0 & 0 & 2 \\ 101 & 119 & 120 & 210 & 219 & 260 & 392 \end{vmatrix}$. $= 0$ since row 1 and row 4 are proportional.

3. (15+10=25 points) Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix}$.

(a) Find A^{-1} using the Gauss-Jordan method.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_3\to R_3]{-R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -3 & 0 & -1 & 1 \end{array}\right] \xrightarrow{\frac{R_3}{3}+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 2/3 & 1/3 \\ 0 & 0 & -3 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{2R_3}{3}+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2/3 & 2/3 \\ 0 & 1 & 0 & -1 & 2/3 & 1/3 \\ 0 & 0 & -3 & 0 & -1 & 1 \end{array}\right] \xrightarrow{\frac{R_3}{3}\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 1 & -2/3 & 2/3 \\ -1 & 2/3 & 1/3 \\ 0 & 1/3 & -1/3 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

(b) Find the first row of A^{-1} using the adjoint matrix method.

$$A^{-1} = \frac{\operatorname{adj} A}{\det A} = \frac{(\text{cofactor } A)^T}{\det A}$$

$$\det A = \begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= -3$$

$$(A^{-1})_{11} = \frac{M_{11}}{-3} = \frac{\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix}}{-3} = 1$$

$$(A^{-1})_{12} = \frac{-M_{21}}{-3} = \frac{-\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix}}{-3} = \frac{-2}{3}$$

$$(A^{-1})_{13} = \frac{M_{31}}{-3} = \frac{\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}}{-3} = \frac{2}{3}$$

4. (6+6+6=18 points) (a) Suppose that the reduced row echelon form of the matrix A is

$$\begin{bmatrix} 0 & \textcircled{1} & 7 & 0 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

How many free parameters does the system of equations $A\mathbf{x} = \mathbf{0}$ have? Why?

2 free parameters, since there are 3 leading 1's and 2 columns without leading 1's.

(b) Suppose that A and B are symmetric $n \times n$ matrices.

- What is the definition of a symmetric matrix?

  $A^T = A$

- Is it always true that AB is symmetric? Prove or give a counterexample. What about $AB + BA$? What about $A - B$?

AB is not necessarily symmetric: example: $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 13 \\ 11 & 21 \end{bmatrix}$

$(AB+BA)^T = (AB)^T + (BA)^T = B^TA^T + A^TB^T = BA + AB = AB + BA$, so $AB+BA$ is symmetric.

$(A-B)^T = A^T - B^T = A - B \Rightarrow A-B$ is symmetric.

(c) Say $A = \begin{bmatrix} l_1 & 0 & 0 \\ * & l_2 & 0 \\ * & * & l_3 \end{bmatrix} \times \begin{bmatrix} u_1 & * & * \\ 0 & u_2 & * \\ 0 & 0 & u_3 \end{bmatrix}$ where $*$ denotes an unknown number, and suppose that $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. Show that at least one of $u_1, u_2, u_3, l_1, l_2, l_3$ must be zero.

Since $A\vec{x} = \vec{0}$ has a nontrivial solution, $\det A = 0$.

but $\det A = \begin{vmatrix} \ell_1 & 0 & 0 \\ * & \ell_2 & 0 \\ * & * & \ell_3 \end{vmatrix} \cdot \begin{vmatrix} u_1 & * & * \\ 0 & u_2 & * \\ 0 & 0 & u_3 \end{vmatrix} = \ell_1\ell_2\ell_3u_1u_2u_3$

So at least one of $u_1, u_2, u_3, \ell_1, \ell_2, \ell_3$ must be 0.

5. (6+6=12 points) Let $A = (a_{ij})$ be the following 6×6 matrix:

$$\begin{bmatrix} 0 & 1 & \textcircled{2} & 0 & 0 & 0 \\ \textcircled{1} & 0 & 0 & 2^{\times} & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 1^{\times} & 0 & 0 & 0 & 0 & \textcircled{2} \\ 0 & 2 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(a) Find all permutations σ with $\sigma(i) = j_i$ such that $a_{1j_1}a_{2j_2}a_{3j_3}a_{4j_4}a_{5j_5}a_{6j_6}$ is not zero. Determine the signatures of these permutations.

$\sigma(1)=3$, $\sigma(2)=1$ and $\sigma(5)=6$

For the rest we have two choices

① $\sigma(1)=3$
$\sigma(2)=1$
$\sigma(3)=2$
$\sigma(4)=4$
$\sigma(5)=6$
$\sigma(6)=5$

Signature:

First method: Cycle type

$\underbrace{(1\ 3\ 2)}_{\text{even}}\underbrace{(5\ 6)}_{\text{odd}} \Rightarrow$ odd

Second method: Inversion count

$3>1,\ 3>2,\ 6>5$: 3 inversions

$\Rightarrow$ odd.

② $\sigma(1)=3$
$\sigma(2)=1$
$\sigma(3)=4$
$\sigma(4)=5$
$\sigma(5)=6$
$\sigma(6)=2$

Signature

First method: Cycle type

$\underbrace{(1\ 3\ 4\ 5\ 6\ 2)}_{\text{odd}} \Rightarrow$ odd

Second method: Inversion count

$3>1,\ 3>2,\ 4>2,\ 5>2,\ 6>2$: 5 inversions

$\Rightarrow$ odd.

(b) Compute $\det(A)$ using the information in part (a).

$$\det A = -a_{13}a_{21}a_{32}a_{44}a_{56}a_{65} - a_{13}a_{21}a_{34}a_{45}a_{56}a_{62}$$

$$= -4 - 32$$

$$= -36$$