A third method for visualizing the twisted cubic is to realize that it also lies on the cylinder $z = x^3$. So it can be viewed as the curve of intersection of the cylinders $y = x^2$ and $z = x^3$. (See Figure 11.)
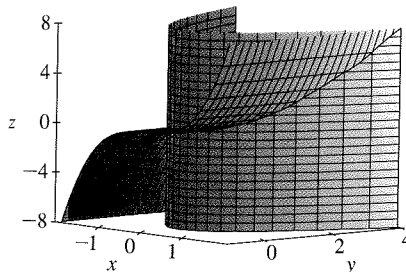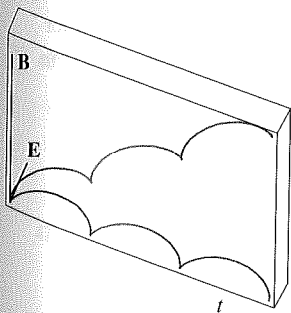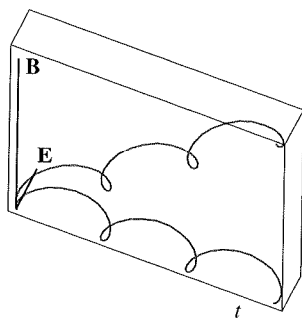
FIGURE 11

Some computer algebra systems provide us with a clearer picture of a space curve by enclosing it in a tube. Such a plot enables us to see whether one part of a curve passes in front of or behind another part of the curve. For example, Figure 13 shows the curve of Figure 12(b) as rendered by the `tubeplot` command in Maple.

We have seen that an interesting space curve, the helix, occurs in the model of DNA. Another notable example of a space curve in science is the trajectory of a positively charged particle in orthogonally oriented electric and magnetic fields **E** and **B**. Depending on the initial velocity given the particle at the origin, the path of the particle is either a space curve whose projection on the horizontal plane is the cycloid we studied in Section 10.1 [Figure 12(a)] or a curve whose projection is the trochoid investigated in Exercise 40 in Section 10.1 [Figure 12(b)].

(a) $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t, t \rangle$

(b) $\mathbf{r}(t) = \langle t - \frac{3}{2} \sin t, 1 - \frac{3}{2} \cos t, t \rangle$

FIGURE 12

Motion of a charged particle in orthogonally oriented electric and magnetic fields
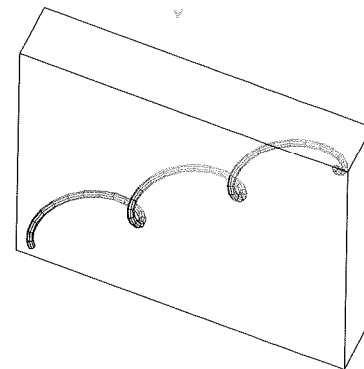
FIGURE 13

For further details concerning the physics involved and animations of the trajectories of the particles, see the following web sites:

- www.phy.ntnu.edu.tw/java/emField/emField.html
- www.physics.ucla.edu/plasma-exp/Beam/

13.1 Exercises

1–2 Find the domain of the vector function.

1. $\mathbf{r}(t) = \left\langle \sqrt{4 - t^2}, e^{-3t}, \ln(t + 1) \right\rangle$

2. $\mathbf{r}(t) = \dfrac{t - 2}{t + 2} \mathbf{i} + \sin t \, \mathbf{j} + \ln(9 - t^2) \mathbf{k}$

3–6 Find the limit.

3. $\displaystyle \lim_{t \to 0} \left(e^{-3t} \mathbf{i} + \frac{t^2}{\sin^2 t} \mathbf{j} + \cos 2t \, \mathbf{k} \right)$

4. $\displaystyle \lim_{t \to 1} \left(\frac{t^2 - t}{t - 1} \mathbf{i} + \sqrt{t + 8} \, \mathbf{j} + \frac{\sin \pi t}{\ln t} \mathbf{k} \right)$

⊞ Graphing calculator or computer required   **1.** Homework Hints available at stewartcalculus.com

5. $\lim\limits_{t \to \infty} \left\langle \dfrac{1 + t^2}{1 - t^2}, \tan^{-1} t, \dfrac{1 - e^{-2t}}{t} \right\rangle$

6. $\lim\limits_{t \to \infty} \left\langle te^{-t}, \dfrac{t^3 + t}{2t^3 - 1}, t \sin \dfrac{1}{t} \right\rangle$

7–14 Sketch the curve with the given vector equation. Indicate with an arrow the direction in which t increases.

7. $\mathbf{r}(t) = \langle \sin t, t \rangle$

8. $\mathbf{r}(t) = \langle t^3, t^2 \rangle$

9. $\mathbf{r}(t) = \langle t, 2 - t, 2t \rangle$

10. $\mathbf{r}(t) = \langle \sin \pi t, t, \cos \pi t \rangle$

11. $\mathbf{r}(t) = \langle 1, \cos t, 2 \sin t \rangle$

12. $\mathbf{r}(t) = t^2 \mathbf{i} + t \mathbf{j} + 2 \mathbf{k}$

13. $\mathbf{r}(t) = t^2 \mathbf{i} + t^4 \mathbf{j} + t^6 \mathbf{k}$

14. $\mathbf{r}(t) = \cos t \, \mathbf{i} - \cos t \, \mathbf{j} + \sin t \, \mathbf{k}$

15–16 Draw the projections of the curve on the three coordinate planes. Use these projections to help sketch the curve.

15. $\mathbf{r}(t) = \langle t, \sin t, 2 \cos t \rangle$

16. $\mathbf{r}(t) = \langle t, t, t^2 \rangle$

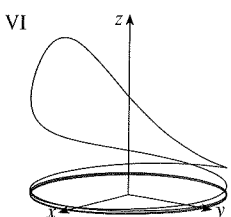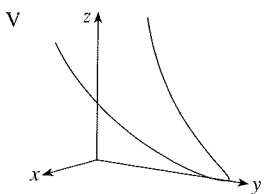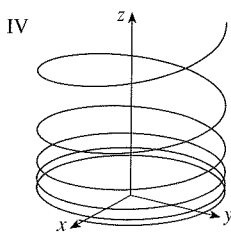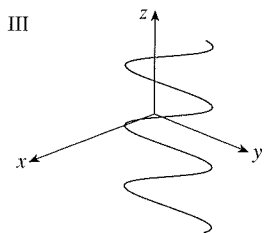**17–20** Find a vector equation and parametric equations for the line segment that joins P to Q.

17. $P(0, 0, 0)$, $Q(1, 2, 3)$

18. $P(1, 0, 1)$, $Q(2, 3, 1)$

19. $P(0, -1, 1)$, $Q\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)$

20. $P(a, b, c)$, $Q(u, v, w)$

21–26 Match the parametric equations with the graphs (labeled I–VI). Give reasons for your choices.

I

II

III

IV

V

VI
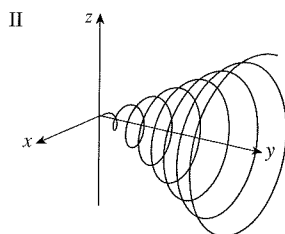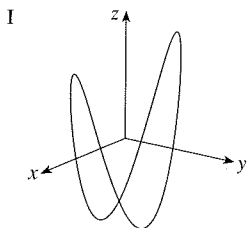
21. $x = t \cos t$, $y = t$, $z = t \sin t$, $t \geqslant 0$

22. $x = \cos t$, $y = \sin t$, $z = 1/(1 + t^2)$

23. $x = t$, $y = 1/(1 + t^2)$, $z = t^2$

24. $x = \cos t$, $y = \sin t$, $z = \cos 2t$

25. $x = \cos 8t$, $y = \sin 8t$, $z = e^{0.8t}$, $t \geqslant 0$

26. $x = \cos^2 t$, $y = \sin^2 t$, $z = t$

---

27. Show that the curve with parametric equations $x = t \cos t$, $y = t \sin t$, $z = t$ lies on the cone $z^2 = x^2 + y^2$, and use this fact to help sketch the curve.

28. Show that the curve with parametric equations $x = \sin t$, $y = \cos t$, $z = \sin^2 t$ is the curve of intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$. Use this fact to help sketch the curve.

29. At what points does the curve $\mathbf{r}(t) = t \, \mathbf{i} + (2t - t^2) \, \mathbf{k}$ intersect the paraboloid $z = x^2 + y^2$?

30. At what points does the helix $\mathbf{r}(t) = \langle \sin t, \cos t, t \rangle$ intersect the sphere $x^2 + y^2 + z^2 = 5$?

31–35 Use a computer to graph the curve with the given vector equation. Make sure you choose a parameter domain and viewpoints that reveal the true nature of the curve.

31. $\mathbf{r}(t) = \langle \cos t \sin 2t, \sin t \sin 2t, \cos 2t \rangle$

32. $\mathbf{r}(t) = \langle t^2, \ln t, t \rangle$

33. $\mathbf{r}(t) = \langle t, t \sin t, t \cos t \rangle$

34. $\mathbf{r}(t) = \langle t, e^t, \cos t \rangle$

35. $\mathbf{r}(t) = \langle \cos 2t, \cos 3t, \cos 4t \rangle$

---

36. Graph the curve with parametric equations $x = \sin t$, $y = \sin 2t$, $z = \cos 4t$. Explain its shape by graphing its projections onto the three coordinate planes.

37. Graph the curve with parametric equations

$$x = (1 + \cos 16t) \cos t$$

$$y = (1 + \cos 16t) \sin t$$

$$z = 1 + \cos 16t$$

Explain the appearance of the graph by showing that it lies on a cone.

38. Graph the curve with parametric equations

$$x = \sqrt{1 - 0.25 \cos^2 10t} \, \cos t$$

$$y = \sqrt{1 - 0.25 \cos^2 10t} \, \sin t$$

$$z = 0.5 \cos 10t$$

Explain the appearance of the graph by showing that it lies on a sphere.

39. Show that the curve with parametric equations $x = t^2$, $y = 1 - 3t, z = 1 + t^3$ passes through the points $(1, 4, 0)$ and $(9, -8, 28)$ but not through the point $(4, 7, -6)$.

40–44 Find a vector function that represents the curve of intersection of the two surfaces.

40. The cylinder $x^2 + y^2 = 4$ and the surface $z = xy$

41. The cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1 + y$

42. The paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$

43. The hyperboloid $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$

44. The semiellipsoid $x^2 + y^2 + 4z^2 = 4, y \geq 0$, and the cylinder $x^2 + z^2 = 1$

45. Try to sketch by hand the curve of intersection of the circular cylinder $x^2 + y^2 = 4$ and the parabolic cylinder $z = x^2$. Then find parametric equations for this curve and use these equations and a computer to graph the curve.

46. Try to sketch by hand the curve of intersection of the parabolic cylinder $y = x^2$ and the top half of the ellipsoid $x^2 + 4y^2 + 4z^2 = 16$. Then find parametric equations for this curve and use these equations and a computer to graph the curve.

47. If two objects travel through space along two different curves, it's often important to know whether they will collide. (Will a missile hit its moving target? Will two aircraft collide?) The curves might intersect, but we need to know whether the objects are in the same position *at the same time.* Suppose the trajectories of two particles are given by the vector functions

$$\mathbf{r}_1(t) = \langle t^2, 7t - 12, t^2 \rangle \qquad \mathbf{r}_2(t) = \langle 4t - 3, t^2, 5t - 6 \rangle$$

for $t \geq 0$. Do the particles collide?

48. Two particles travel along the space curves

$$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle \qquad \mathbf{r}_2(t) = \langle 1 + 2t, 1 + 6t, 1 + 14t \rangle$$

Do the particles collide? Do their paths intersect?

49. Suppose $\mathbf{u}$ and $\mathbf{v}$ are vector functions that possess limits as $t \to a$ and let c be a constant. Prove the following properties of limits.

(a) $\lim\limits_{t \to a} [\mathbf{u}(t) + \mathbf{v}(t)] = \lim\limits_{t \to a} \mathbf{u}(t) + \lim\limits_{t \to a} \mathbf{v}(t)$

(b) $\lim\limits_{t \to a} c\mathbf{u}(t) = c \lim\limits_{t \to a} \mathbf{u}(t)$

(c) $\lim\limits_{t \to a} [\mathbf{u}(t) \cdot \mathbf{v}(t)] = \lim\limits_{t \to a} \mathbf{u}(t) \cdot \lim\limits_{t \to a} \mathbf{v}(t)$

(d) $\lim\limits_{t \to a} [\mathbf{u}(t) \times \mathbf{v}(t)] = \lim\limits_{t \to a} \mathbf{u}(t) \times \lim\limits_{t \to a} \mathbf{v}(t)$

50. The view of the trefoil knot shown in Figure 8 is accurate, but it doesn't reveal the whole story. Use the parametric equations

$$x = (2 + \cos 1.5t) \cos t$$
$$y = (2 + \cos 1.5t) \sin t$$
$$z = \sin 1.5t$$

to sketch the curve by hand as viewed from above, with gaps indicating where the curve passes over itself. Start by showing that the projection of the curve onto the xy-plane has polar coordinates $r = 2 + \cos 1.5t$ and $\theta = t$, so r varies between 1 and 3. Then show that z has maximum and minimum values when the projection is halfway between $r = 1$ and $r = 3$.

When you have finished your sketch, use a computer to draw the curve with viewpoint directly above and compare with your sketch. Then use the computer to draw the curve from several other viewpoints. You can get a better impression of the curve if you plot a tube with radius 0.2 around the curve. (Use the `tubeplot` command in Maple or the `tubecurve` or `Tube` command in Mathematica.)

51. Show that $\lim\limits_{t \to a} \mathbf{r}(t) = \mathbf{b}$ if and only if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < |t - a| < \delta \quad \text{then} \quad |\mathbf{r}(t) - \mathbf{b}| < \varepsilon$$

13.2 Derivatives and Integrals of Vector Functions

Later in this chapter we are going to use vector functions to describe the motion of planets and other objects through space. Here we prepare the way by developing the calculus of vector functions.

Derivatives

The **derivative** $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined in much the same way as for real-valued functions:

$$\boxed{1} \qquad \frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h}$$