METU-NCC
Mathematics Group

	Calculus with Analytic Geometry Final Exam	
Code : MATH 119 Acad.Year : 2011 Semester : Spring Coord. : S.D./H.T. Date : 05.06.2012 Time : 9.00 Duration : 150 minutes	Last Name : Name : Stud. No : Dept. : Sec. No : Signature : 6 Questions on 6 Pages Total 100 Points	

Q1	Q2	Q3	Q4	Q5	Q6		

Q.1 (3 + 3 + 3 + 6 = 15 pts) This question has 4 INDEPENDENT parts !!

(a) WITHOUT calculating the coefficients, write the FORM of the Partial Fraction Decomposition of $\dfrac{1}{(x-1)^2(5x^2+2x+10)(x^2+4)^2}$ $\Delta<0$. $\Delta<0$.

$$\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{5x^2+2x+10}+\frac{Ex+F}{x^2+4}+\frac{Gx+H}{(x^2+4)^2}$$

(b) Find y' if $y=(x^2+3)^{119}(e^{x^2}+x+1)^{2011}(1+\sin x)^{2012}$.

$$\ln y = 119\ln(x^2+3)+2011\ln(e^{x^2}+x+1)+2012\ln(1+\sin x)$$

$$\frac{y'}{y}=119\frac{2x}{x^2+3}+2011\frac{e^{x^2}\cdot 2x+1}{e^{x^2}+x+1}+2012\frac{\cos x}{1+\sin x}$$

$$y'=y\cdot(\qquad\downarrow\qquad)$$

(c) Find y' if $y=x^{(x^2+\ln x+1)}$.

$$\ln y=(x^2+\ln x+1)\ln x$$

$$\frac{y'}{y}=\left(2x+\frac{1}{x}\right)\ln x+(x^2+\ln x+1)\cdot\frac{1}{x}$$

(d) Find $f'(10)$ and $(f^{-1})'(0)$, if $f(x)=\int_2^x e^{t^2}\,dt$.

$$f\circ f^{-1}(x)=x.$$

$$f'(x)=e^{x^2}.$$

$$f'(f^{-1}(x))\cdot(f^{-1})'(x)=1$$

$$f'(10)=e^{100}$$

$$(f^{-1})'(0)=\frac{1}{f'(\underbrace{f^{-1}(0)}_{2})}=\frac{1}{f'(2)}=e^{4}.$$

$$(f'(x)>0 \Rightarrow f \text{ is } 1\text{-}1)$$

Q.2 ($4 \times 5 = 20$ pts) Evaluate the following limits:

(a) $\lim_{x\to 0^+} \frac{\ln x}{1+(\ln x)^2}$ $\overset{LH}{=} \lim_{x\to 0^+} \frac{\cancel{1/x}}{2\ln x \cdot \cancel{\frac{1}{x}}} = \boxed{0.}$

(b) $\lim_{x\to 0} (1-2x)^{4/x}$ $= e^{\lim_{x\to 0} \frac{4\ln(1-2x)}{x}} = e^{\lim_{x\to 0} 4 \frac{\frac{-2}{1-2x}}{1}}$

$= e^{4\cdot(-2)} = \boxed{e^{-8}}$

(c) $\lim_{x\to 0} \left(\cot x - \frac{1}{x}\right)$ $= \lim_{x\to 0} \frac{x\cos x - \sin x}{x\sin x} = \lim_{x\to 0} \frac{\cancel{\cos x} - x\sin x - \cancel{\cos x}}{\sin x + x\cos x}$

$\frac{\cos x}{\sin x} - \frac{1}{x}$ $= \lim_{x\to 0} \frac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0.$

(d) $\lim_{x\to 0} \frac{\ln(x+1)}{x\tan x - \cos x}$ $= \frac{0}{-1} = 0.$

Q.3 (5 × 3 = 15 pts) Given $y = f(x) = \frac{e^x}{x}$.

(a) Write down the domain of f, and find its asymptotes.

$D(f) = \mathbb{R} \setminus \{0\}$.

$\lim_{x\to 0^+} \frac{e^x}{x} = \infty$ $x = 0$ vertical asymp.

$\lim_{x\to \infty} \frac{e^x}{x} = \infty$

$\lim_{x\to 0^-} \frac{e^x}{x} = -\infty$.

$\lim_{x\to -\infty} \frac{e^x}{x} = 0$. $\boxed{y=0 \text{ hor. asymp.}}$

(b) Find intervals of increase and decrease.

$f'(x) = \frac{e^x}{x} - \frac{e^x}{x^2} = \frac{e^x}{x^2}\left(x - 1\right)$

f is inc on $(1, \infty)$

dec on $(-\infty, 0) \cup (0, 1)$.

		0	1
f'	$-$	$-$	$+$
f''	$-$	$+$	$+$

(c) Find local maximum and minimum values of f if there is any.

f has local min at $x = 1$.

no local max.

(d) Find intervals of concavity. Is there any inflection points?

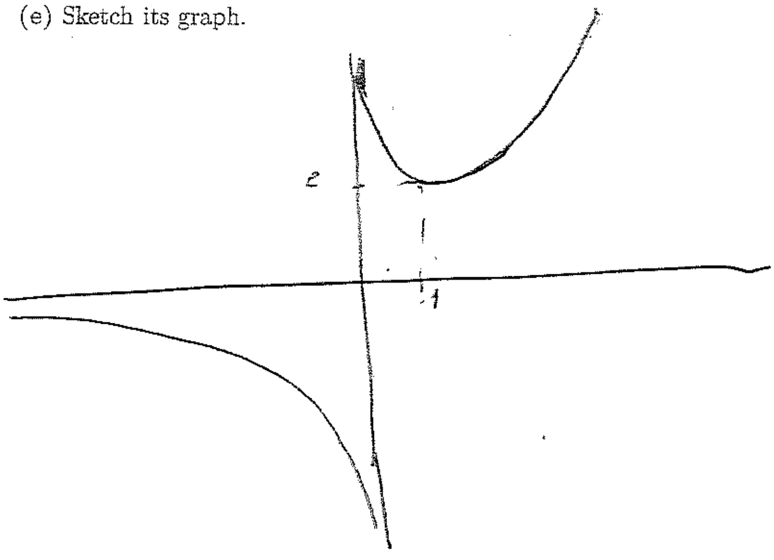
$f''(x) = \frac{e^x}{x} - \frac{e^x}{x^2} - \frac{e^x}{x^2} + \frac{2e^x}{x^3} = \frac{e^x}{x^3}\left(x^2 - 2x + 2\right)$ > 0.

f is con. down on $(-\infty, 0)$

up on $(0, \infty)$

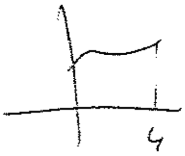
no inf. pnts since $0 \notin D(f)$.

(e) Sketch its graph.



Q.4 ($4 \times 5 = 20$ pts) Let C be the CURVE given by the graph of $f(x) = e^{\sin x}$ between $x = 0$ and $x = 4$. Let R be the REGION between C and the x-axis. WITHOUT evaluating the integrals:

(a) Find the ARCLENGTH of C.



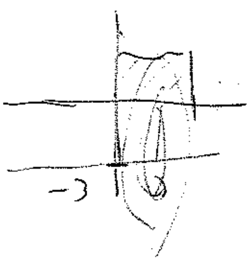
$$\int_0^4 \sqrt{1+f'(x)^2}\,dx = \int_0^4 \sqrt{1+(e^{\sin x}\cdot\cos x)^2}\,dx.$$

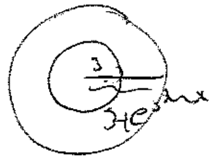
$$f'(x) = e^{\sin x}\cdot\cos x$$

(b) Find the VOLUME of the solid obtained by rotating R about the x-axis.

$$\int_0^4 \pi\,(e^{\sin x})^2\,dx$$

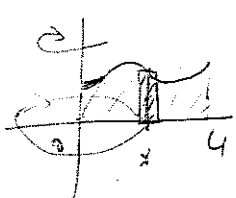
(c) Find the VOLUME of the solid obtained by rotating R about the line $y = -3$.

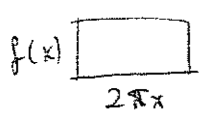




$$\int_0^4 \pi\left((3+e^{\sin x})^2 - 9\right)dx$$

(d) Find the VOLUME of the solid obtained by rotating R about the y-axis. (HINT: Use Shell Method)





$$\int_0^4 2\pi x\, e^{\sin x}\,dx.$$

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Q.5 ($5 \times 4 = 20$ pts) Evaluate the following integrals:

(a) $\int x\, 3^{x^2+1}\, dx = \frac{1}{2}\int 3^u\, du = \frac{1}{2}\frac{3^u}{\ln 3} + C = \frac{1}{2}\frac{3^{x^2+1}}{\ln 3} + C$

$x^2 + 1 = u$

$2x\,dx = du$

(b) $\int \sin^5 x \cos^9 x\, dx = -\int (1-u^2)^2 u^9\, du = -\int u^9 - 2u^{11} + u^{13}\, du$

$\cos x = u.$

$-\sin x\, dx = du.$

$= -\frac{u^{10}}{10} + \frac{2u^{12}}{12} - \frac{u^{14}}{14} + c.$

$\sin^4 x = (1 - \cos^2 x)^2$

$u^9(1-u^2)^2 = (1 - 2u^2 + u^4)u^9$

$= u^9 - 2u^{11} + u^{13}$

(c) $\int \frac{\ln x}{\sqrt{x}}\, dx = 2\sqrt{x}\ln x - \int \frac{2\sqrt{x}}{\cancel{x}\sqrt{x}}\, dx = 2\sqrt{x}\ln x - 4\sqrt{x} + c.$

$\ln x = u \quad \frac{1}{x}dx = du$

$\frac{dx}{\sqrt{x}} = dv \quad 2\sqrt{x} = v$

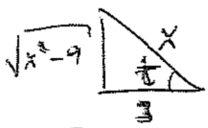
(d) $\int \frac{x^3+1}{x^3-x^2}\, dx = \int dx + \int \frac{2}{x-1}dx - \int \frac{dx}{x} - \int \frac{dx}{x^2}$

$= x + 2\ln|x-1| - \ln x + \frac{1}{x} + c$

$\frac{x^3 - x^2}{x^3 - x^2} + \frac{x^2+1}{x^3-x^2}$

$\frac{x^2+1}{x^3-x^2} = \frac{2}{x-1} + \frac{-1}{x} - \frac{1}{x^2}$

(e) $\int \frac{9}{x^2\sqrt{x^2-9}}\,dx = \int \frac{\cancel{9}}{\cancel{9}\sec^2 t\,\cancel{\tan t}}\,3\sec t\,\cancel{\tan t}\,dt = \int 3\cos t\,dt$



$= 3\sin t + C = \frac{3.\sqrt{x^2-9}}{x} + C.$

$\frac{x}{3} = \sec t \qquad \sqrt{x^2-9} = 3\tan x$

$\frac{dx}{3} = \sec t\tan t\,dt$

Q.6 ($2 \times 5 = 10$ pts)

(a) Evaluate the integral $\int_1^\infty \frac{\ln x}{x^3}\,dx$, if it is convergent.

$\frac{\ln x}{x^3} < \frac{x}{x^3} \leq \frac{1}{x^2} \qquad \int_1^\infty \frac{1}{x^2}\,dx$ conv. $\Rightarrow \int_1^\infty \frac{\ln x}{x^3}\,dx$ is conv.

$\ln x = u \Rightarrow \frac{dx}{x} = du$

$\frac{dx}{x^3} = dv \Rightarrow \frac{1}{-2x^2} = v.$

$\int_1^\infty \frac{\ln x}{x^3}\,dx = \frac{1}{-2x^2}\ln x\Big|_1^t + \int_1^t \frac{dx}{2x^3} = \frac{-1}{4x^2}\Big|_1^t$

$= \lim_{t\to\infty} \overset{\to 0}{\frac{\ln t}{-2t^2}} + \overset{=0}{\frac{\ln 1}{2.1}} - \overset{\to 0}{\frac{1}{4t^2}} + \frac{1}{4} = \frac{1}{4}.$

(b) Determine whether the integral

$$\int_1^\infty \frac{1+\sin\sqrt{x}}{\sqrt{x}+x^3}\,dx$$

is convergent or divergent.

$\frac{1+\overset{\leq 1}{\sin\sqrt{x}}}{\sqrt{x}+x^3} \leq \frac{2}{\underbrace{\sqrt{x}+x^3}_{\geq x^3}} \leq \frac{2}{x^3}$

$\int_1^\infty \frac{2}{x^3}\,dx$ is conv by p-test

$\Rightarrow \int_1^\infty \frac{\overset{\geq 0}{(1+\sin\sqrt{x})}}{\underset{\neq 0}{(\sqrt{x}+x^3)}}\,dx$ is conv.