M E T U – N C C
Mathematics Group

	Calculus with Analytic Geometry Second Midterm Exam	

Code : MATH 119 Acad.Year : 2010-2011 Semester : Spring Instructors : A.D./H.T./B.W. Date : 30.04.2011 Time : 15.30 Duration : 120 minutes	Last Name : Name : KEY Stud. No : Dept. : Sec. No : Signature : **6** Questions on **6** Pages Total **100** Points

1	2	3	4	5	6		

Q.1 (4 × 5 = 20 pts) Find the following derivatives:

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{(x-1)^{2/3}\sqrt{x^2+1}}{x^4\sin x}$ (Use logarithmic differentiation) Since $\ln(y) =$

$= \frac{2}{3}\ln(x-1) + \frac{1}{2}\ln(x^2+1) - 4\ln(x) - \ln(\sin(x))$,

we have $y'\frac{1}{y} = \frac{2}{3(x-1)} + \frac{x}{x^2+1} - \frac{4}{x} - \cot(x)$ or

$y' = y\left(\frac{2}{3(x-1)} + \frac{x}{x^2+1} - \frac{4}{x} - \cot(x)\right)$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\arcsin\left[\ln(2^x)\right] = \left[\arcsin(\ln(2)x)\right]' =$

$= \dfrac{\ln(2)}{\sqrt{1-(\ln(2)x)^2}}$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_x^{x^2} t^2\tan t\,\mathrm{d}t = \frac{d}{dx}\int_0^{x^2} t^2\tan(t)\,dt - \frac{d}{dx}\int_0^{x} t^2\tan(t)\,dt$

$= x^4\tan(x^2)\cdot 2x - x^2\tan(x)$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\ln x)^{\ln x}$ Since $\ln(y) = \ln(x)\ln(\ln(x))$

$y'\frac{1}{y} = \frac{1}{x}\ln(\ln(x)) + \ln(x)\frac{1}{\ln(x)}\frac{1}{x}$

$y' = y\,\frac{1}{x}\left(\ln(\ln(x)) + 1\right)$

Q.2 (4 × 5 = 20 pts) Evaluate the following integrals:

(a) $\displaystyle\int \frac{x^9}{\sqrt{x^5+2}}\,dx = \left| \begin{array}{l} u = x^5+2 \\ du = 5x^4 dx \end{array} \right| = \frac{1}{5}\int \frac{(u-2)\,du}{\sqrt{u}} =$

$= \frac{1}{5}\int \sqrt{u}\,du - \frac{2}{5}\int \frac{du}{\sqrt{u}} = \frac{2}{15}u^{3/2} - \frac{4}{5}\sqrt{u} + C$

$= \frac{2}{15}(x^5+2)^{3/2} - \frac{4}{5}(x^5+2)^{1/2} + C$

(b) $\displaystyle\int_0^{1/2} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \left| \begin{array}{ll} u = \arcsin(x) & , x=0 \Rightarrow u=0 \\ du = \frac{dx}{\sqrt{1-x^2}} & , x=\frac{1}{2} \Rightarrow u=\pi/6 \end{array} \right| =$

$= \displaystyle\int_0^{\pi/6} u\,du = \frac{u^2}{2}\Big|_0^{\pi/6} = \frac{\pi^2}{72}$

(c) $\displaystyle\int_{-1}^{1} x^8 \sin x\,dx = 0$, for $f(-x) = -f(x)$ with

$f(x) = x^8 \sin(x)$, $-1 \le x \le 1$.

(d) $\displaystyle\int \frac{\sec(\ln x)\tan(\ln x)}{x}\,dx = \left| \begin{array}{l} u = \ln(x) \\ du = \frac{dx}{x} \end{array} \right| = \int \sec(u)\tan(u)\,du$

$= \sec(u) + C = \sec(\ln(x)) + C$

Q.3 (5 × 3 = 15 pts) Consider the function $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$.

(a) Write down its domain. Is the line $x = 0$ a vertical asymptote? Why?

domain$(f) = (0, \infty)$. Since $\lim_{x \to 0^+} f(x) = \infty$, the line $x = 0$ is a vertical asymptote.

(b) Find intervals of increase and decrease.

$$f'(x) = \frac{x-1}{2x^{3/2}} \Rightarrow C(f) = \{1\}$$

$f'(x) > 0$ on $(1, \infty) \Rightarrow f$ increases

$f'(x) < 0$ on $(0, 1) \Rightarrow f$ decreases.

(c) Find local maximum and minimum points if there is any.

Based on (b), we conclude that $f(x)$ has a local minimum at $x = 1$, and $f(1) = 2$.

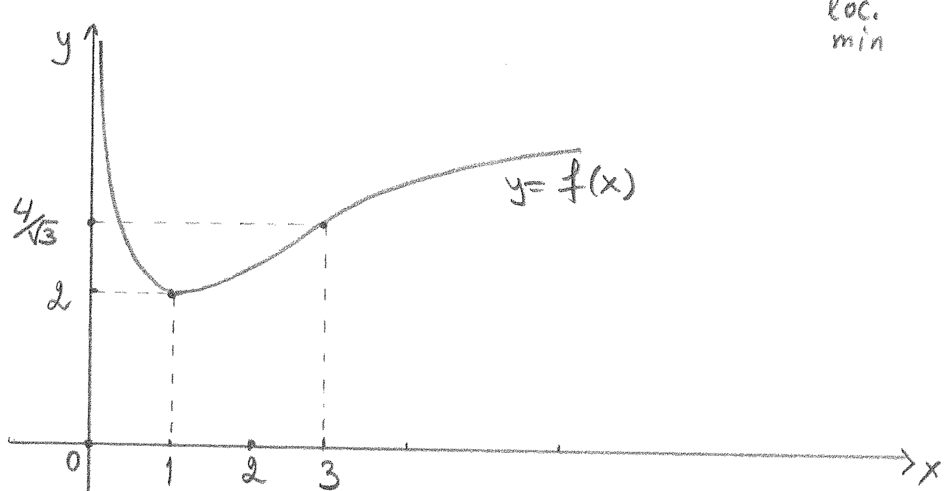
(d) Find intervals of concavity. Is there any inflection points?

$$f''(x) = \frac{3-x}{4x^{5/2}} \Rightarrow C(f') = \{3\}.$$

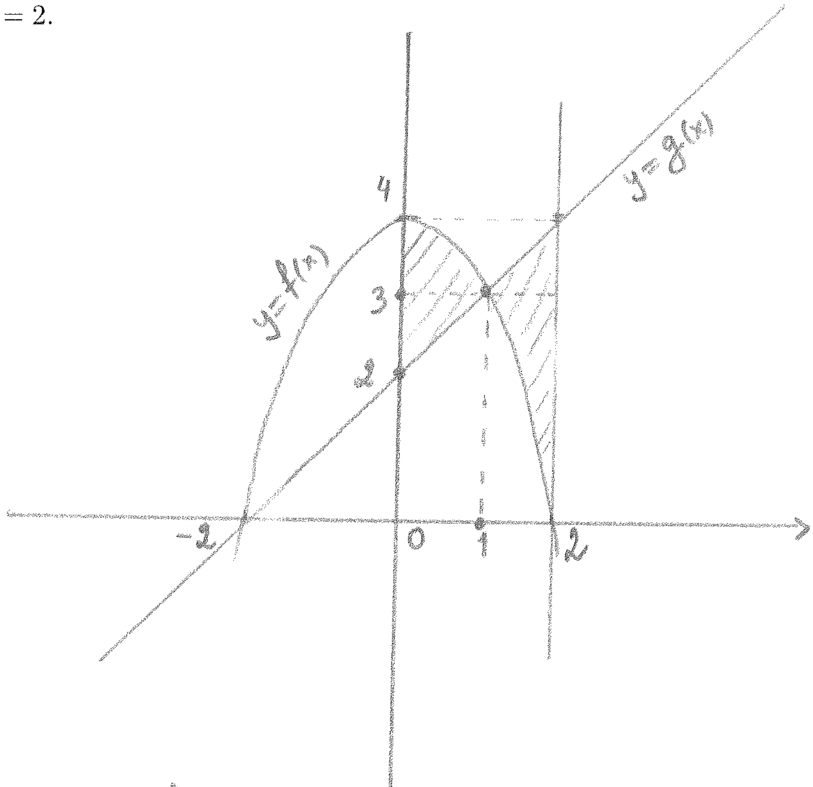
$x = 3$ is an inflection point.

	0	1	3
x	$0 < x < 1$	$1 < x < 3$	$3 < x$
sign $f'(x)$	$-$	$+$	$+$
sign $f''(x)$	$+$	$+$	$-$
behavior $f(x)$	dec. conc. up	incr. conc up	incr conc. down
		loc. min	inf.

(e) Sketch its graph.



Q.4 (15 pts) Let $f(x) = 4 - x^2$ and $g(x) = 2 + x$. Calculate the **area** (finding its numerical value) of the region bounded by f and g between the lines $x = 0$ and $x = 2$.



$$\left(f(x)=g(x)\right) \Longleftrightarrow \left(x^2+x-2=0\right) \Longleftrightarrow (x=-2,1)$$

We have

$$\text{area} = \int_0^1 [f(x)-g(x)]dx + \int_1^2 [g(x)-f(x)]dx =$$

$$= \int_0^1 [(4-x^2)-(2+x)]dx + \int_1^2 [(2+x)-(4-x^2)]dx =$$

$$= \int_0^1 (2-x-x^2)\,dx + \int_1^2 (x^2+x-2)\,dx =$$

$$= \left(2x - \frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 + \left(\frac{x^3}{3} + \frac{x^2}{2} - 2x\right)\Big|_1^2$$

$$= 2 - \frac{1}{2} - \frac{1}{3} + \frac{8}{3} + 2 - 4 - \frac{1}{3} - \frac{1}{2} + 2$$

$$= 3.$$

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Q.5 (20 pts) Find the **minimal** surface area of a cylindrical can with volume 16π. <u>**Hint**</u>. The surface area and volume are given, respectively, by $A(r,h) = 2\pi r^2 + 2\pi rh$ and $V = \pi r^2 h$, where r is the radius of the base (and the top) of the can, and h denotes its height.

Since $V = \pi r^2 h = 16\pi$, it follows that $h = \frac{16}{r^2}$, and the total area turns out to be a function of r only:

$$A(r) = 2\pi r^2 + 2\pi r \frac{16}{r^2} = 2\pi\left(r^2 + \frac{16}{r}\right),$$

$r > 0$. Note that

$$A'(r) = 2\pi\left(2r - \frac{16}{r^2}\right) = 4\pi r\left(1 - \frac{8}{r^3}\right),$$

that is, $r = 2$ is a critical point.

Using SDT, we derive that

$$A''(r) = 2\pi\left(2 + \frac{32}{r^3}\right), \quad A''(2) > 0,$$

the function takes on minimum value when $r = 2$.

So, $A_{min} = A(2) = 2\pi\left(4 + \frac{16}{2}\right) = 24\pi.$

Q.6 (10 pts) Use the mean value theorem (MVT) to show that $\ln x < x - 1$ for $x > 1$.

Consider the interval $[1, x]$ $(x>1)$ and the continuous function $f(t) = \ln(t)$ on it. Since the function $f(t)$ is differentiable on $(1, x)$, we could apply MVT. Namely,

$$\frac{\ln(x)}{x-1} = \frac{f(x) - f(1)}{x-1} = f'(c) \text{ for some}$$

$c \in (1, x)$ But $f'(c) = \frac{1}{c} < 1$.

Hence

$$\frac{\ln(x)}{x-1} = \frac{1}{c} < 1 \text{ or } \ln(x) < x - 1.$$