M E T U – N C C
Mathematics Group

	Calculus with Analytic Geometry First Midterm Exam	

Code : MATH 119 Acad.Year : 2011 Semester : Fall Coord. : S.D/I.U/H.T.	Last Name : Name : Stud. No : Dept. : Sec. No : Signature :
Date : 03.12.2011 Time : 9.40 Duration : 120 minutes	**8** Questions on **6** Pages Total **100** Points

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8		

Q.1 **(5 + 5 = 10 pts)** This problem has two INDEPENDENT parts.

(a) Calculate $F'(x)$, if $F(x) = \displaystyle\int_{\sqrt{x}}^{x^2} \frac{\cos(u^2)}{u^3+1}\,du + x^2$.

$$F(x) = -\int_0^{\sqrt{x}} \frac{\cos(-u^2)}{u^3+1}\,.du + \int_0^{x^2} \frac{\cos(-u^2)}{u^3+1}\,.du + x^2$$

$$F'(x) = -\frac{\cos(x)}{x^{3/2}+1}\cdot\frac{1}{2\sqrt{x}} + \frac{\cos(x^4)}{x^6+1}\,.\,2x + 2x$$

(b) Find a function f and a number α such that $8+\displaystyle\int_{\alpha}^{\sqrt{x}} t f(t)\,dt = x^{3/2}$.

Differentiate both sides:

$$\sqrt{x}\, f(\sqrt{x})\cdot\frac{1}{2\sqrt{x}} = \frac{3}{2}\sqrt{x}$$

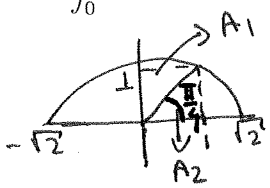
$$f(\sqrt{x}) = 3\sqrt{x} \Rightarrow f(x) = 3x$$

For $x = \alpha^2$:

$$8+0 = \alpha^3 \Rightarrow \alpha = 2.$$

Q.2 ($4 \times 5 = 20$ **pts**) Evaluate the following integrals:

(a) $\int_0^1 \sqrt{2-x^2}\,\mathrm{d}x$ **Hint:** Interpret as an area.



$A_1 = \dfrac{\pi(\sqrt{2})^2}{8}$ $A_2 = \dfrac{1}{2}$

$A_1 + A_2 = \dfrac{\pi}{4} + \dfrac{1}{2}$

(b) $\int \sqrt{x}(x-1)^2\,\mathrm{d}x$ $= \int \sqrt{x}\,(x^2 - 2x + 1)\,dx$

$= \int (x^{5/2} - 2x^{3/2} + \sqrt{x})\,dx$

$= \frac{2}{7}x^{7/2} - \frac{4}{5}x^{5/2} + \frac{2}{3}x^{3/2} + C$

* (c) $\int_{-\pi}^{\pi} [x^8 \sin(x^5) + x^2]\,\mathrm{d}x$ $= \int_{-\pi}^{\pi} x^8 \sin(x^5)dx + \int_{-\pi}^{\pi} x^2 dx = 0 + \frac{2}{3}\pi^3$

$f(x) = x^8(\sin(x^5))$

$f(-x) = x^8 \sin(-x^5) = -x^8 \sin(x^5) = -f(x)$, so $f(x)$ is odd

Hence, $\int_{-\pi}^{\pi} x^8 . \sin(x^5)dx = 0$, $\int_{-\pi}^{\pi} x^2 dx = 2 . \int_0^{\pi} x^2 dx = 2 . \frac{x^3}{3}\Big|_0^{\pi} = 2 . \frac{\pi^3}{3}$

(d) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x$ $= \int 2 . \sin(u)\,du = -2\cos u + C$

$= -2\cos(\sqrt{x}) + C$

Let $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} . dx$

* (e) $\int_0^1 \frac{x^3}{\sqrt{x^2+2}}\,\mathrm{d}x$ $= \int_2^3 \frac{1}{2} . \frac{(u-2)}{u^{1/2}} . du = \frac{1}{2}\int_2^3 (u^{1/2} - 2u^{-1/2})\,du$

Let $u = x^2 + 2$

$du = 2x . dx$

& $x^2 = u - 2$

$= \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 4u^{1/2}\Big|_2^3\right)$

$= \frac{1}{2}\left(\left(\frac{2}{3}3^{3/2} - 4 . \sqrt{3}\right) - \left(\frac{2}{3}2^{3/2} - \sqrt{2}\right)\right)$

Q.3 (10 pts) Find the area of the region enclosed (bounded) by the curves

$$f(x) = x^3 + 4x^2 - x + 3 \quad \text{and} \quad g(x) = x^2 + 3x + 3.$$

Find Intersection points

$$x^3+4x^2-x+3 = x^2+3x+3$$

$$x^3+3x^2-4x=0$$

$$x(x^2+3x-4)=0$$

$$x(x+4)(x-1)=0$$

		-4		0		1	
$f(x)-g(x)$	$-$	ϕ	$+$	ϕ	$-$	ϕ	$+$

$$A = \int_{-4}^{0} [(x^3+4x^2-x+3) - (x^2+3x+3)]dx + \int_0^1 [(x^2+3x+3)-(x^3+4x^2-x+3)]d$$

$$A = \frac{1}{4}x^4 + x^3 - 2x^2 \Big|_{-4}^{0} + -\frac{1}{4}x^4 - x^3 + 2x^2 \Big|_0^1$$

$$A = 0-(64-64-32) + \left(-\frac{1}{4}-1+2\right) - 0$$

$$A = 32 + \frac{3}{4}$$

Q.4 (4 × 5 = 20 pts) Given $f(x) = \dfrac{x-1}{(x+1)^2} = \dfrac{1}{x+1} - \dfrac{2}{(x+1)^2}$.

(a) Write down the domain of the function, and find its asymptotes.

$\mathrm{Dom}(f) = \mathbb{R} - \{-1\}$

$\lim_{x\to -1^+} f(x) = -\infty = \lim_{x \to -1^-} f(x) \rightarrow$ Vertical asymptote at $x = -1$

$\lim_{x\to\infty} f(x) = 0 = \lim_{x\to -\infty} f(x) \rightarrow$ Horizontal asymptote $y = 0$

(b) Find intervals of increase and decrease.

$$f'(x) = -\frac{1}{(x+1)^2} + \frac{4}{(x+1)^3} = \frac{-x+3}{(x+1)^3}$$

		-1		3	
$f'(x)$	$-$		$+$	0	$-$

Local max

(c) Find local maximum and minimum points if there is any.

No local minimum

Local maximum at $x = 3$.

(d) Find intervals of concavity. Is there any inflection points?

$$f''(x) = \frac{2}{(x+1)^3} - \frac{12}{(x+1)^4} = \frac{2x-10}{(x+1)^4}$$

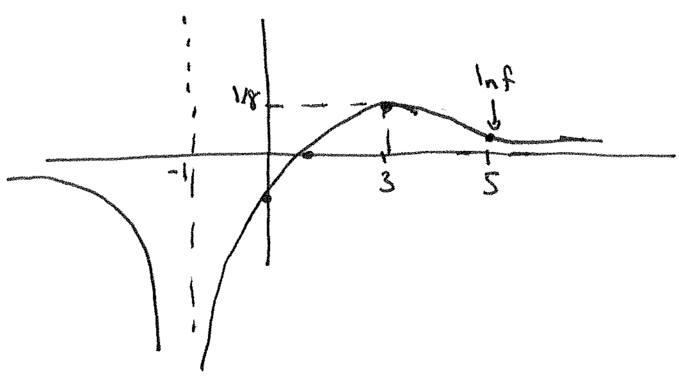
		-1		5	
$f''(x)$	$-$		$-$		$+$

At $x = 5$, there's an inflection point.

(e) Sketch its graph.

$f(0) = -1$ $\quad f(3) = \frac{1}{8}$

$f(1) = 0$



Last Name:

Name:

Q.5 **(4 + 4 = 8 pts)** An object moves along a line with a velocity $v(t) = t^2 - 5t + 6$ at time t.

(a) Find the displacement (net change of position) of the object during the time period $0 \leq t \leq 4$.

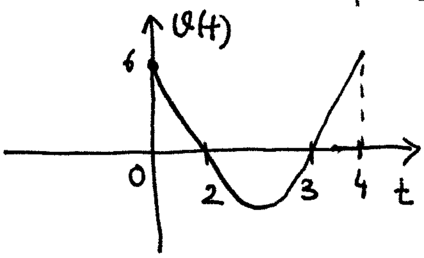
$$\int_0^4 (t^2 - 5t + 6) = \frac{t^3}{3} - \frac{5}{2}t^2 + 6t \Big|_0^4$$

$$= \frac{64}{3} - 40 + 24$$

(b) Find the distance travelled during this time period.

$$\int_0^4 |t^2 - 5t + 6|\,dt = \int_0^2 (t^2 - 5t + 6)\,dt - \int_2^3 (t^2 - 5t + 6)\,dt + \int_3^4 (t^2 - 5t + 6)$$

$$t^2 - 5t + 6 = (t-2)(t-3)$$



$$= \left(\frac{t^3}{3} - \frac{5}{2}t^2 + 6t \Big|_0^2\right) - \left(\frac{t^3}{3} - \frac{5}{2}t^2 + 6t \Big|_2^3\right) + \left(\frac{t^3}{3} - \frac{5}{2}t^2 + 6t\right.$$

$$= \left(\frac{8}{3} - 10 + 12\right) - \left(9 - \frac{45}{2} + 18\right) + \left(\frac{8}{3} - 10 + 12\right)$$

$$+ \left(\frac{64}{3} - 40 + 24\right) - \left(9 - \frac{45}{2} + 18\right)$$

Q.6 **(7 pts)** Evaluate the Riemann sum for $f(x) = \cos(\pi x) + x$ on the interval $x \in [-1, 2]$ dividing it into $n = 6$ subintervals of equal width. Take the sample points x_k^* to be the RIGHT endpoints of each subinterval.

$$\Delta x = \frac{2-(-1)}{6} = \frac{1}{2}$$

$x_0 = -1$

$x_1 = -1 + \Delta x$

$\vdots$

$x_6 = 2$

$x_1^* = x_1$

$x_2^* = x_2$

$\vdots$

$x_6^* = x_6$

$$R_6 = \frac{1}{2}\left(f(-\tfrac{1}{2}) + f(0) + f(\tfrac{1}{2}) + f(1) + f(3/2) + f(2)\right)$$

$$R_6 = \frac{1}{2}\left[(0 - \tfrac{1}{2}) + (1 + 0) + (0 + \tfrac{1}{2}) + (-1 + 1) + (0 + \tfrac{3}{2}) + (1 + 2)\right]$$

$$R_6 = \frac{11}{4}$$

Q.7 (15 pts) A rectangular box has dimensions x, $3-x$ and $6-x$. Find the value of x which **maximizes** the **volume** of the box.

$V(x) = x(3-x)(6-x) = x^3 - 9x^2 + 18x$

$V'(x) = 3x^2 - 18x + 18 = 0$

$x^2 - 6x + 6 = 0$

Crit points : $x_1 = \frac{6+\sqrt{12}}{2}$ $\quad x_2 = \frac{6-\sqrt{12}}{2}$

$x_1 = 3+\sqrt{3}$ $\quad x_2 = 3-\sqrt{3}$

Since dimensions have to be positive, $0 < x < 3$.
So we have only critical point $(3-\sqrt{3})$ in the domain.

$V''(x) = 2x-6$ & $V''(x) < 0$ for all $x \in (0,3)$.
So $3-\sqrt{3}$ is a global max of $V(x)$ on $(0,3)$.

Q.8 (10 pts) Show that the equation

$$x^5 + x^3 + x + 1 - \frac{1}{4}\sin(\pi x) = 0$$

has EXACTLY ONE real root on the interval $x \in (-1, 0)$.

$f(x) = x^5 + x^3 + x + 1 - \frac{1}{4}\sin(\pi x)$

$f(-1) = -1-1-1+1 < 0$
$f(0) = 1 > 0$
} By Intermediate Value Theorem $f(x)$ has a zero in $(-1,0)$.

$f'(x) = 5x^4 + 3x^2 + 1 - \frac{\pi}{4}\cos(\pi x)$

Since $\frac{\pi}{4} < 1$, $-1 < \frac{\pi}{4}\cos(x) < 1$. Hence

$f'(x) > 0$ for all x. Then by Rolle's Theorem (or mean value theorem) $f(x)$ can't have more than 1 zeroes