MATH219, Homework I Solutions Fall 2010

1) $\frac{dy}{dx} = \frac{y}{x} + \frac{1}{2}\sqrt{1-(\frac{y}{x})^2}$ (Algebraiclly homogeneous)

Let $u = \frac{y}{x} \to \frac{dy}{dx} = u + x\frac{du}{dx} \Rightarrow u + x\frac{du}{dx} = u + \frac{1}{2}\sqrt{1-u^2}$

• $\frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\frac{dx}{x} \Rightarrow \arcsin u = \frac{1}{2}\ln x + C$

$\Rightarrow u = \sin(\ln\sqrt{x} + C)$

$\Rightarrow y = x\sin(\ln\sqrt{x} + C)$

•• $y(1) = 0 \Rightarrow \sin C = 0 \Rightarrow C = 0$

$\therefore$ $y = y(x) = x\sin(\ln\sqrt{x})$

(c) What is the **interval of definition** of the solution in Part **(b)**?

$x > 0$ or $x \in (0, \infty)$

2) This is a homogenous equation since

$$\frac{dy}{dx} = \frac{\frac{xy}{x^2} + 2\frac{y^2}{x^2}}{\frac{xy}{x^2} + \frac{x^2}{x^2}} = \frac{v + 2v^2}{v+1}$$

where $v = \frac{y}{x}$. The left hand side is $v + x\frac{dv}{dx}$. So

$$x\frac{dv}{dx} = \frac{v+2v^2}{v+1} - v = \frac{v^2}{v+1}$$

$$\int \frac{v+1}{v^2}dv = \int \frac{dx}{x}$$

$$\ln|v| - \frac{1}{v} = \ln|x| + c$$

$$\ln|\frac{y}{x}| - \frac{x}{y} = \ln|x| + c$$

(1)

3)

Question 2 (10 + 10 = 20) Solve the following differential equations:

(a) $\dfrac{dy}{dx} = \dfrac{x^2 - 2y}{x}$

$$\frac{dy}{dx} + \frac{2}{x} y = x \quad (\text{Linear}) \quad \mu(x) = e^{\int \frac{2}{x} dx} = e^{\ln x^2} = x^2$$

$$\frac{d}{dx}(x^2 y) = x^3 \to x^2 y = \frac{1}{4} x^4 + C$$

$$y = y(x) = \frac{1}{4} x^2 + \frac{C}{x^2}$$

$$ty' - 2y = t^4 \cos(t^2), \quad y(\sqrt{\pi}) = 1.$$

$$y' - \frac{2}{t} y = t^3 \cos(t^2), \qquad \mu = e^{\int -\frac{2}{t} dt} = e^{-2\ln t} = \frac{1}{t^2}$$

5) $$\Rightarrow \frac{1}{t^2} y' - \frac{2}{t^3} y = t \cos(t^2)$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{t^2} y\right] = t\cos(t^2) \Rightarrow \frac{1}{t^2} y = \int t \cos(\underbrace{t^2}_{u}) dt + c = \frac{1}{2} \sin(t^2) + c$$

$$\Rightarrow y = ct^2 + \frac{t^2}{2} \sin(t^2)$$

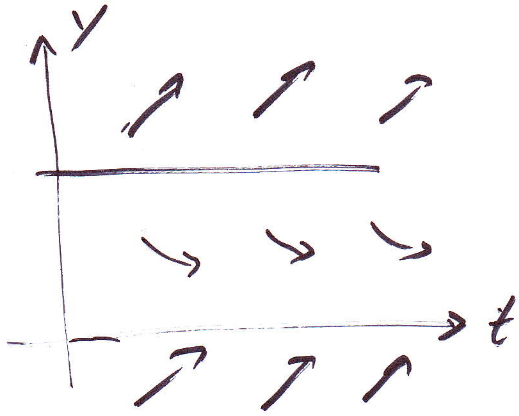
$$1 = y(\sqrt{\pi}) = c \cdot \pi + \frac{\pi}{2} \overset{0}{\cancel{\sin \pi}} \Rightarrow \boxed{c = \frac{1}{\pi}}$$

$$\boxed{y = \frac{1}{\pi} t^2 + \frac{1}{2} t^2 \sin(t^2)}$$

(2)

5) a) $y' = y^2 - 9y = y(y-9)$

$y=0$, $y=9$ equilibrium points

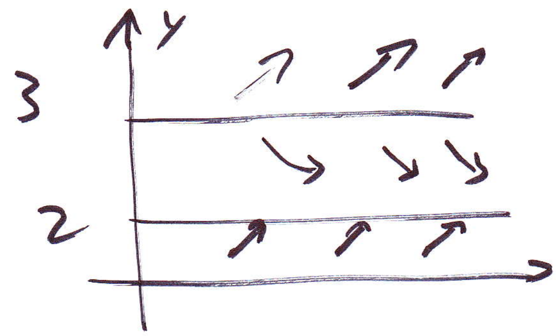


unstable

stable

b) $y' = y^2 - 5y + 6 = (y-2)(y-3)$

$y=2$, $y=3$ equilibrium points

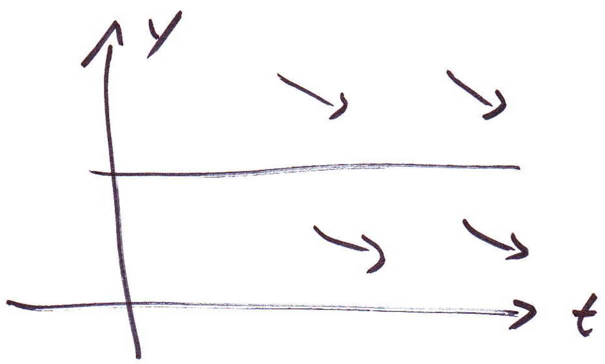


unstable

stable

c) $y' = -y^2 + 6y - 9 = -(y-3)^2$

$y = 3$ equilibrium point



semi-stable

1. (20p) Use the substitution $\boxed{v = \frac{1}{y}}$ to solve $\boxed{xy' - (1+2x)y = y^2, \quad x > 0}$.

8) $y = \frac{1}{v} \Rightarrow y' = -\frac{1}{v^2}v'$, $v \neq 0$

Then, $-x\frac{1}{v^2}v' - (1+2x)\frac{1}{v} = \frac{1}{v^2}$ Bernoulli

$\Rightarrow -xv' - (1+2x)v = 1$, $v \neq 0$

$\Rightarrow v' + \frac{1+2x}{x}v = -\frac{1}{x}$

$$\mu(x) = e^{\int \frac{(1+2x)}{x}dx} = e^{\ln x + 2x} = xe^{2x}$$

So, $\frac{d}{dx}\left[xe^{2x}v\right] = -e^{2x}$

$\Rightarrow xe^{2x}v = -\frac{1}{2}e^{2x} + c$

$\Rightarrow v = c\frac{1}{x}e^{-2x} - \frac{1}{2x} = \frac{1}{y}$

$$\Rightarrow y = \frac{2x}{2ce^{-2x} - 1}, \quad x > 0.$$

Note also that $y \equiv 0$ is a particular solution.

7)

$V = y^{1-6} = y^{-5}$, $y = V^5$ Bernoulli

$y' = -5y^{-6}v'$

$x(-5y^{-6}v') + v^{-5} = x^3$

$\frac{dv}{dx} - \frac{5}{x}v \neq 5x^2$, $\mu = e^{-\int\frac{5}{x}} = x^{-5}$

$\frac{d}{dx}(x^{-5}v) = -5x^{-3}$

$v(x) = \frac{5}{2}x^3 + cx^5$

$$y(x) = \left(\frac{1}{x^3(cx^2 + \frac{5}{2})}\right)^{-1/5}$$ (6)

8) (a) $V(t) = 50 + 2t$.

(b) $\dfrac{dQ}{dt} = 3 - \dfrac{Q}{50+2t}$

(c) This is a first order linear equation. $\mu(x) = e^{\int \frac{1}{50+2t}dt} = (50+2t)^{\frac{1}{2}}$ is an integrating factor. So

$$\begin{aligned} \frac{d(Q\cdot(50+2t)^{\frac{1}{2}})}{dt} &= 3(50+2t)^{\frac{1}{2}} \\ Q\cdot(50+2t)^{\frac{1}{2}} &= (50+2t)^{\frac{3}{2}} + c \end{aligned} \tag{2}$$

Since $Q(0) = 0$, $c = -50^{\frac{3}{2}}$.

(d) The tank fills up when $t = 25$. Then $Q(25) = \frac{100^{\frac{3}{2}} - 50^{\frac{3}{2}}}{10} \simeq 64.64$.

(5)