M E T U
Northern Cyprus Campus

Math 260 Linear Algebra Midterm Exam II 05.12.2013		
Last Name Name : Student No	Dept./Sec.: Time : 17: 40 Duration : 70 *minutes*	Signature
5 QUESTIONS ON 4 PAGES	KEY	TOTAL 100 POINTS

1	2	3	4	5				

Q1 (15=5+10 p.) Consider $T \in \mathfrak{L}(\mathbb{R}^3)$, $T(x, y, z) = (2y + z, x + z, x + 2y)$.

a) Show that T is an isomorphism.

It suffices to prove that $\ker(T) = \{0\}$. But

$\vec{v} = (x,y,z) \in \ker(T)$ iff $\begin{cases} 2y + z = 0 \\ x + z = 0 \\ x + 2y = 0 \end{cases}$. In this case, $z = -x$

$\begin{cases} -x + 2y = 0 \\ x + 2y = 0 \end{cases} \Rightarrow y = x = z = 0.$

b) Find the inverse transformation T^{-1}.

Solve the linear system $\begin{cases} 2y + z = a \\ x + z = b \\ x + 2y = c \end{cases}$. We have

$\begin{cases} x - 2y = b - a \\ x + 2y = c \end{cases} \Rightarrow 2x = b - a + c \Rightarrow x = \dfrac{b-a+c}{2}$. But

$2y = c - x = \dfrac{a-b+c}{2} \Rightarrow y = \dfrac{a-b+c}{4}$. Finally, $z = b - x$

$= b - \dfrac{b-a+c}{2} = \dfrac{a+b-c}{2}$. Hence

$$T^{-1}(a,b,c) = \left(\frac{-a+b+c}{2}, \frac{a-b+c}{4}, \frac{a+b-c}{2} \right)$$

for all $(a,b,c) \in \mathbb{R}^3$.

Q2 (25=15+10 p.) Consider $T : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^2$, $T(p(x)) = \left(3p(0), \int_0^2 p(x)\,dx\right)$.

a) Find the matrix $M_{(e,f)}(T)$ of T relative to the pair of bases $e = (1, 2x, 3x^2, 4x^3)$ for $\mathcal{P}_3(\mathbb{R})$, and $f = ((1,1),(1,0))$ for $\mathbb{R}^2$.

Note that $T(1) = (3,2) = 2f_1 + f_2$, $T(2x) =$
$= (0,4) = 4f_1 - 4f_2$, $T(3x^2) = (0,8) = 8f_1 - 8f_2$,
$T(4x^3) = (0,16) = 16f_1 - 16f_2$.

Therefore

$$M_{(e,f)}(T) = \begin{bmatrix} 2 & 4 & 8 & 16 \\ 1 & -4 & -8 & -16 \end{bmatrix}$$

b) Find a basis for the subspace $\ker(T)$ in $\mathcal{P}_3(\mathbb{R})$.

Take $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \in \ker(T)$. Then
$a_0 = p(0) = 0$ and $\int_0^2 p(x)dx = a_1 \frac{x^2}{2}\Big|_0^2 + a_2 \frac{x^3}{3}\Big|_0^2 + a_3 \frac{x^4}{4}\Big|_0^2$
$= 2a_1 + \frac{8}{3}a_2 + 4a_3 = 0.$

Hence $\ker(T) = \{ a_0 = 0, \ 2a_1 + \frac{8}{3}a_2 + 4a_3 = 0 \}$
and $f = (f_1, f_2)$ with
$f_1 = (0, 4, -3, 0)$, $f_2 = (0, 0, -3, 2)$
is a basis for $\ker(T)$.

Q3 (25=5+20 p.) Consider $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(x, y, z) = (x + z, z - y)$.

a) Find the matrix $M_{(e,f)}(T)$ with respect to the standard bases e and f for $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

$$M_{(e,f)}(T) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

b) Using the Change of Base Formula, find the matrix $M_{(e',f')}(T)$ with respect the pair of bases $e' = (e'_1, e'_2, e'_3)$ and $f' = (f'_1, f'_2)$, where $e'_1 = (2, 0, 0)$, $e'_2 = (0, 1, -1)$, $e'_3 = (0, 2, -1)$, and $f'_1 = (1/2, -1/2)$, $f'_2 = (1/2, 1/2)$.

We have $M_{(e',f')}(T) = M_{(f,f')}(\mathbb{1})\, M_{(e,f)}(T)\, M_{(e',e)}(\mathbb{1})$.

Note that $M_{(e',e)}(\mathbb{1}) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix}$, and

$f_1 = (1, 0) = f'_1 + f'_2$, $f_2 = (0, 1) = -f'_1 + f'_2$, that is,

$M_{(f,f')}(\mathbb{1}) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. It follow that

$$M_{(e',f')}(T) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 2 & -3 & -4 \end{bmatrix}.$$

Q4 (25 p.) Let $V = \mathbb{R}^4$ and $W = \{x + y + z - w = 0\}$ be a 3-dimensional subspace in V. Consider the projection $P \in \mathfrak{L}(V)$ onto the subspace W parallel to the vector $\mathrm{v} = (1, 0, 0, 0)$. Find the matrix $M_{(e,e)}(P)$ of P relative to the standard basis e for V.

Choose a basis f_1, f_2, f_3 for the subspace W: $f_1 = (1, -1, 0, 0)$, $f_2 = (0, 1, -1, 0)$, $f_3 = (0, 0, 1, 1)$. Then $f = (f_1, f_2, f_3, f_4)$ with $f_4 = \vec{v}$ is a basis for V and

$$M_{(f,f)}(P) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

But

$e_1 = f_4 \Rightarrow P(e_1) = 0$; $e_2 = -f_1 + f_4 \Rightarrow P(e_2) = -f_1$;
$e_3 = -f_1 - f_2 + f_4 \Rightarrow P(e_3) = -f_1 - f_2$; $e_4 = f_1 + f_2 + f_3 - f_4$
$\Rightarrow P(e_4) = f_1 + f_2 + f_3$. Hence

$$M_{(e,e)}(P) = \begin{bmatrix} 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q5 (10 p.) Let V be a vector space (of any dimension), and $T \in \mathfrak{L}(V)$ a nilpotent linear transformation such that $T^k = 0$ and $T^{k-1} \neq 0$ for $k > 1$. Show that $\ker(T) \neq \{0\}$ and $\operatorname{im}(T) \neq V$.

Since $T^{k-1} \neq 0$, we have $T^{k-1}(v) \neq 0$ for some $v \in V$. Then $T(T^{k-1}(v)) = T^k(v) = 0$, that is, $T^{k-1}(v) \in \ker(T)$.

If $\operatorname{im}(T) = V$, then $\forall y \in V, \exists x \in V, y = T(x)$ $\Rightarrow T^{k-1}(y) = T^k(x) = 0$, that is, $T^{k-1} = 0$, a contradiction.